ALAGAPPA UNIVERSITY

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KARAIKUDI - 630 003

DIRECTORATE OF DISTANCE EDUCATION

M.Sc

31141

GRAPH THEORY

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Chapter 1

GRAPHS

Unit- I

1.1 GRAPHS

Definition 1.1 *A* ***graph*** *is an ordered triple* $G = (V(G), E(G), I_G)$*, where* $V(G)$ *is a nonempty set,* $E(G)$ *is a set disjoint from* $V(G)$*, and* I_G *is an "incidence" map that associates with each element of* $E(G)$*, an unordered pair of elements(same or distinct) of* $V(G)$*.*
Elements of $V(G)$ *are called the* ***vertices****(or nodes or points) of*G*, and elements of* $E(G)$ *are called the* ***edges****(or lines) of* G*. If, for the edge* e *of* G*,* $I_G(e) = \{u, v\}$*, we write* $I_G(e) = uv$*.*

Example 1.1 *If* $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$*,* $E(G) = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ *and* I_G *is given by* $I_G(e_1) = \{v_1, v_5\}$*,*$I_G(e_2) = \{v_2, v_3\}$*,*$I_G(e_4) = \{v_2, v_5\}$*,*$I_G(e_5) = \{v_2, v_5\}$*,*$I_G(e_6) = \{v_3, v_3\}$*, then* $(V(G), E(G), I_G)$ *is a graph.*

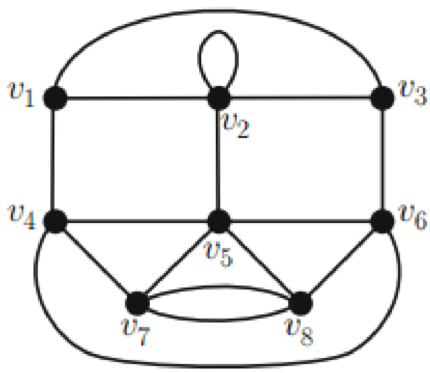


Figure 1.1:

Definition 1.2 *If $I_G(e) = \{u, v\}$, then the vertices u and v are called the* ***end vertices*** *or* ***ends*** *of the edge e. Each edge is said to* ***join*** *its ends; in this case we say that e is* ***incident*** *with each one of its ends. Also, the vertices u and v are then incident with e. A set of two or more edges of a graph G is called a set of* ***multiple*** *or* ***parallel edges*** *if they have the same ends. If e is the only edge with end vertices u and v, we write $e = uv$. An edge for which the two ends are the same is called a* ***loop*** *at the common vertex.*
A vertex u is a ***neighbor*** *of v in G, if uv is an edge of G, and $u \neq v$. The set of all neighbors of v is the* ***open neighborhood*** *of v or the* ***neighbor set*** *of v, and is denoted by $N(v)$; the set $N[v] = N(v) \cup \{v\}$ is the* ***closed neighborhood*** *of v in G. When G must be explicit, these open and closed neighborhoods are denoted by $N_G(v)$ and $N_G[v]$, respectively.*
Vertices u and v are ***adjacent*** *to each other in G if, and only if, there is an edge of G with u and v as its ends. Two distinct edges e and f are said to be* ***adjacent*** *if, and only if, they have a common end vertex.*

Definition 1.3 *A graph is* ***simple*** *if it has no loops and no multiple edges. Thus, for a simple graph G, the incidence function I_G is one-to-one. Hence, an edge of a simple graph is identified with the pair of its ends. A simple graph therefore may be considered as an unordered pair $(V(G), E(G))$, where $V(G)$ is a non-empty set and $E(G)$ is a set of unordered pairs of elements of $V(G)$(each edge of the graph being identified with the pair of its ends).*

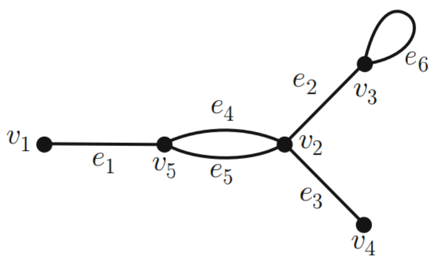


Figure 1.2:

Example 1.2 *In the above graph(1.2) the edge $e_3 = v_2v_4$, edges e_4 and e_5 form multiple edges, e_6 is a loop at v_3, $N(v_2) = \{v_3, v_4, v_5\}, N(v_3) = \{v_2\}, N[v_2] = \{v_2, v_3, v_4, v_5\}$ and $N[v_2] = N(v_2) \cup \{v_2\}$. Further, v_2 and v_5 are adjacent vertices and e_3 and e_4 are adjacent edges.*

Definition 1.4 *A graph is called* ***finite*** *if both $V(G)$ and $E(G)$ are finite. A graph that is not finite is called* ***infinite****.*

Notation 1.1 *We denote by $n(G)$ and $m(G)$ the number of vertices and edges of the graph G, respectively. The number $n(G)$ is called the* ***order*** *of G and $m(G)$ is called the* ***size*** *of G.*

Definition 1.5 *A graph is said to be **labeled**, if its n vertices are distinguished from one another by labels such as $v_1, v_2, \ldots, v_n$.*

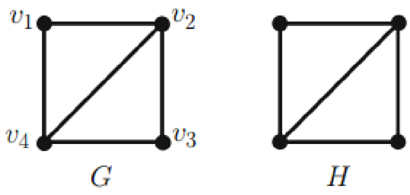


Figure 1.3: A labeled graph G and an unlabeled graph H

Definition 1.6 *A simple graph G is said to be **complete** if every pair of distinct vertices of G are adjacent in G. Any two complete graphs each on a set of n vertices are isomorphic; each such graph is denoted by K_n.*

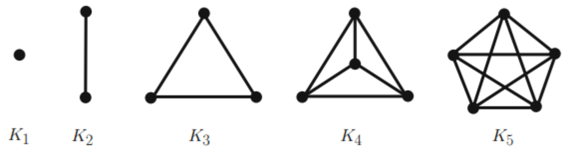


Figure 1.4: Some complete graphs

Note 1.1 *A simple graph with n vertices can have at most $\binom{n}{2} = \frac{n(n-1)}{2}$ edges. K_n has the maximum number of edges among all simple graphs with n vertices. Thus, for a simple graph G with n vertices, we have $0 \le m(G) \le n(n-1)/2$.*

Definition 1.7 *A graph is **trivial** if its vertex set is singleton and it contains no edges.*

Definition 1.8 *Let G be a simple graph. Then the **complement** G^c of G is defined by taking $V(G^c) = V(G)$ and making two vertices u and v adjacent in G^c if, and only if, they are nonadjacent in G. It is clear that G^c is also a simple graph and that $(G^c)^c = G$.*

Note 1.2 *If $|V(G)| = n$, then clearly, $|E(G)| + |E(G^c)| = |E(K_n)| = n(n-1)/2$.*

Definition 1.9 *A simple graph G is called **self-complementary** if $G \cong G^c$.*

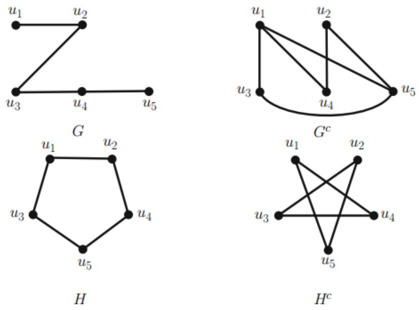


Figure 1.5: Two simple graphs and their complements

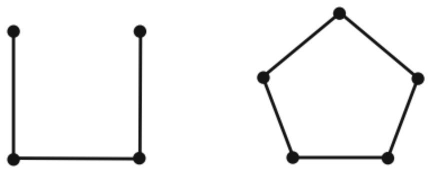


Figure 1.6: Self-complementary graphs

1.2 Subgraphs

Definition 1.10 *A graph H is called a* ***subgraph*** *of G if $V(H) \subset V(G)$, $E(H) \subset E(G)$, and I_H is the restriction of I_G to $E(H)$. If H is a subgraph of G, then G is said to be a* ***supergraph*** *of H. A subgraph H of a graph G is a* ***proper subgraph*** *of G if either $V(H) \neq V(G)$ or $E(H) \neq E(G)$.*

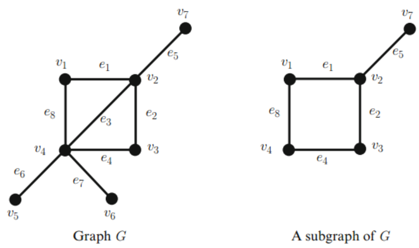


Figure 1.7: A subgraph of G

Definition 1.11 *A subgraph H of G is said to be an* ***induced subgraph*** *of G if each edge of G having its ends in $V(H)$ is also an edge of H. A subgraph H of G is a* ***spanning subgraph*** *of G, if $V(H) = V(G)$. The induced subgraph of G with vertex set $S \subset V(G)$ is called the* ***subgraph of G induced by S*** *and is denoted by $G[S]$.*

Definition 1.12 *A* ***clique*** *of G is a complete subgraph of G. A clique of G is a* ***maximal clique*** *of G if it is not properly contained in another clique of G.*

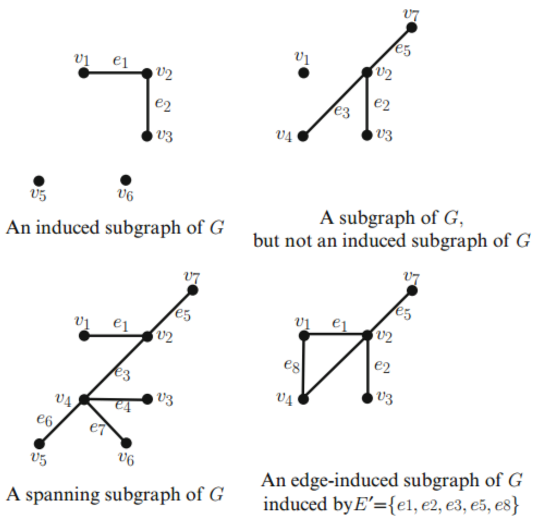


Figure 1.8:

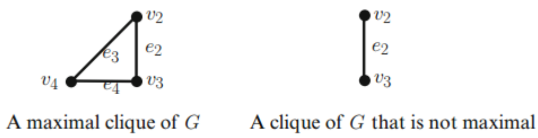


Figure 1.9:

Definition 1.13 *Let G be a graph, S a proper subset of the vertex set V, and E' a subset of E. The subgraph $G[V-S]$ is said to be obtained from G by the* ***deletion*** *of S. This subgraph is denoted by $G-S$.*
The spanning subgraph of G with the edge set E/E' is the subgraph obtained from G by deleting the edge subset E'. This subgraph is denoted by $G-E'$.

Note 1.3 *When a vertex is deleted from G, all the edges incident to it are also deleted from G, whereas the deletion of an edge from G does not affect the vertices of G.*

1.3 Graph Isomorphism

Definition 1.14 *Let $G = (V(G), E(G), I_G)$ and $H = (V(H), E(H), I_H)$ be two graphs. A graph* ***isomorphism*** *from G to H (written $G \cong H$) is a pair (ϕ, θ), where $\phi : V(G) \to V(H)$ and $\theta : E(G) \to E(H)$ are bijections with the property that $I_G(e) = \{u, v\}$ if, and only if, $I_H(\theta(e)) = \{\phi(u), \phi(v)\}$.*

Note 1.4 *If (ϕ, θ) is a graph isomorphism, the pair of the inverse mappings (ϕ^{-1}, θ^{-1}) is also a graph isomorphism. Also note that the bijection ϕ satisfies the condition*

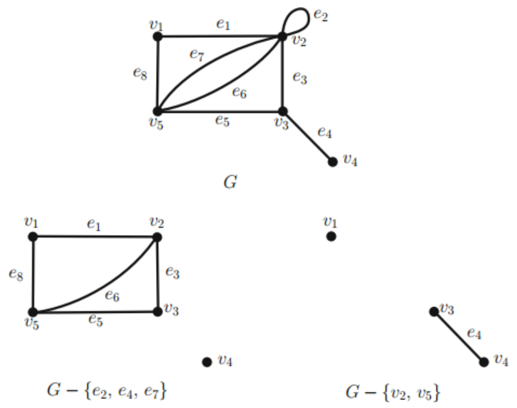


Figure 1.10: Deletion of vertices and edges from a graph G

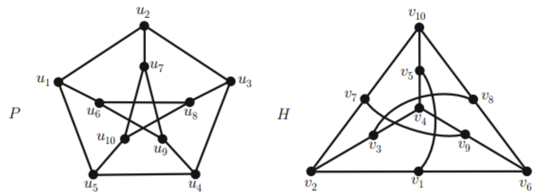


Figure 1.11: Isomorphic graphs

that u and v are end vertices of an edge e of G if, and only if, $\phi(u)$ and $\phi(v)$ are end vertices of the edge $\theta(e)$ in H.

Definition 1.15 *If graphs G and H are simple, a bijection $\phi : V(G) \to V(H)$ such that u and v are adjacent in G if, and only if, $\phi(u)$ and $\phi(v)$ are adjacent in H induces a bijection $\theta : E(G) \to E(H)$ satisfying the condition that $I_G(e) = \{u, v\}$ if, and only if, $I_H(\theta(e)) = \{\phi(u), \phi(v)\}$.*
Hence ϕ itself is referred to as an isomorphism in the case of simple graphs G and H. Thus if G and H are simple graphs, an isomorphism from G to H is a bijection $\phi : V(G) \to V(H)$ such that u and v are adjacent in G if, and only if, $\phi(u)$ and $\phi(v)$ are adjacent in H.

1.4 Incidence and adjacency matrices

Definition 1.16 *Let G be a graph with n vertices, namely $v_1, v_2, \ldots, v_n$. The **adjacency matrix** of G, with respect to these n vertices of G, is the $n \times n$ matrix $A(G) = (a_{ij})$ where the (i, j)th entry a_{ij} is the number of edges joining the vertex v_i to the vertex v_j.*

Definition 1.17 *Suppose that G has n vertices, namely $v_1, v_2, \ldots, v_n$ and t edges, listed as $e_1, e_2, \ldots, e_t$. The **incidence matrix** of G, with respect to these particular* *listing of the vertices and edges of G, is the $n \times t$ matrix $M(G) = (m_{ij})$ where m_{ij} is the number of times that the vertex v_i is incident with the edge e_j, i.e.,*

$$m_{ij} = \begin{cases} 0 \text{ if } v_i \text{ is not an end of } e_j \\ 1 \text{ if } v_i \text{ is an end of the non-loop } e_j \\ 2 \text{ if } v_i \text{ is an end of the loop } e_j. \end{cases}$$

1.5 Vertex degrees

Definition 1.18 *Let G be a graph and $v \in V$. The number of edges incident at v in G is called the **degree**(or **valency**) of the vertex v in G and is denoted by $d_G(v)$, or simply $d(v)$ when G requires no explicit reference. A loop at v is to be counted twice in computing the degree of v.*

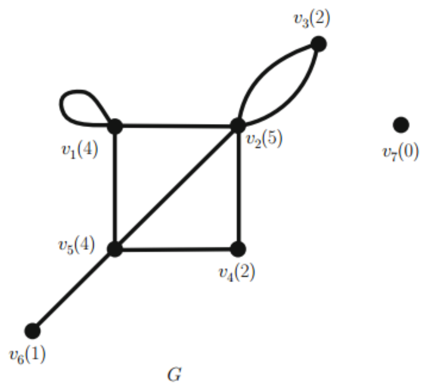


Figure 1.12: Degrees of vertices of a graph G

Notation 1.2 *The minimum(respectively, maximum) of the degrees of the vertices of a graph G is denoted $\delta(G)$ or δ(respectively, $\Delta(G)$ or Δ).*

Definition 1.19 *A graph G is called **k-regular**, if every vertex of G has degree k. A graph is said to be **regular** if it is k-regular for some nonnegative integer k. In particular, a 3-regular graph is called **cubic graph**.*

Definition 1.20 *A spanning 1-regular subgraph of G is called a **1-factor** or a **perfect matching** of G.*

Definition 1.21 *A vertex of degree 0 is known as an **isolated vertex** of G. A vertex of degree 1 is called a **pendant vertex** of G, whereas the unique edge of G incident to such a vertex of G is a pendant edge of G. A sequence formed by the degrees of vertices of G is called a **degree sequence** of G.*

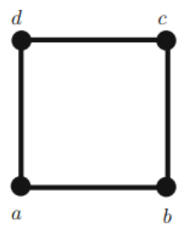


Figure 1.13: Degrees of vertices of a graph G

Theorem 1.1 *The sum of the degrees of the vertices of a graph is equal to twice the number of its edges.*

Proof 1.1 *If $e = uv$ is an edge of G, e is counted once while counting the degrees of each of u and v(even when $u = v$). Hence each edge contributes 2 to the sum of the degrees of the vertices. Thus the m edges of G contributes $2m$ to the degree sum.*

Corollary 1.1 *In any graph G, the number of vertices of odd degree is even.*

Proof 1.2 *Let V_1 and V_2 be the subsets of vertices of G with odd and even degrees, respectively. By theorem7.1,*

$$2m(G) = \sum_{v\in V} d_G(v) = \sum_{v\in V_1} d_G(v) + \sum_{v\in V_2} d_G(v).$$

As $2m(G)$ and $\sum_{v\in V_2} d_G(v)$ are even, $\sum_{v\in V_1} d_G(v)$ is even. Since for each $v \in V_1$, $d_G(v)$ is odd, $|V_1|$ must be even.

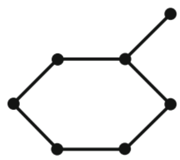
Definition 1.22 *A sequence of nonnegative integers $d = (d_1, d_2, \ldots, d_n)$ is called **graphical** if there exists a simple graph whose degree sequence is d.*

Example 1.3 *The sequence $d = (7, 6, 3, 3, 2, 1, 1, 1)$ is not graphical, even though each term of d is a nonnegative integer and the sum of the terms is even. Indeed, if d were graphical, there must exits a simple graph G with eight vertices whose degree sequence is d. Let v_0 and v_1 be the vertices of G whose degrees are 7 and 6, respectively. Since, G is simple, v_0 is adjacent to another five vertices. This means that in $V - \{v_0, v_1\}$ there must be at least five vertices of degree at least 2. But this not the case.*

Exercise 1.1 *(1) Let G and H be simple graphs and let $\phi : V(G) \to V(H)$ be a bijection such that $uv \in E(G)$ implies that $\phi(u)\phi(v) \in E(H)$. Show, by means of an example, that ϕ need not be an isomorphism from G to H.*

(2) Find the complement of the following simple graph.

(3) Show that if G and H are isomorphic graphs, then each pair of corresponding vertices of G and H have the same degree.



(4) *Let* $d = (d_1, d_2, \ldots, d_n)$ *be the degree sequence of a graph, and* r *be any positive integer. Show that* $\sum_{i=1}^{n} d_i^r$ *is even.*

(5) *Prove that in any group of* n *persons*$(n \geq 2)$*, there are at least two with the same number of friends.*

(6) *Draw the graphs having the following matrices as their adjacency matrices.*

$$(a) \quad \begin{array}{ccccc} 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}$$

Notes:

Chapter 2

WALK & CYCLE

Unit- II

2.1 Walk

Definition 2.1 *A **Walk** in a graph G is an alternating sequence $W : v_0e_1v_1e_2v_2 \dots e_nv_n$ of vertices and edges beginning and ending with vertices in which v_{i-1} and v_i are the ends of e_i; v_0 is the **origin** and v_n is the **terminus** of W. The walk W is said to join v_0 and v_n; it is also referred to as a $v_0 - v_n$ walk.*
*If the graph is simple, a walk is determined by the sequence of its vertices. The walk is **closed** if $v_0 = v_n$ and is **open** otherwise.*

2.2 Path and Cycle

Definition 2.2 *A walk is called a **trial** if all the edges appearing in the walk are distinct. It is called a **path** if all the vertices are distinct. Thus a path in G is automatically a trial in G.*

Definition 2.3 *A **cycle** is a closed trial in which the vertices are all distinct. The **length** of a walk is the number of edges in it. A walk of length zero consists of just a single vertex.*

Definition 2.4 *A graph that is a cycle of length n is denoted by C_n. P_n denotes a path on n vertices. In particular, C_3 is often referred to as a **triangle**, C_4 as a **square**, and C_5 as a **pentagon**. If $P = v_0e_1v_1e_2v_2 \dots e_nv_n$ is a path, then $P^{-1} = v_ne_nv_{n-1}e_{n-1}v_{n-2} \dots v_1e_1v_0$ is also a path and P^{-1} is called the **inverse** of the path P. The subsequence $v_ie_{i+1}v_{i+1} \dots e_jv_j$ of P is called the $v_i - v_j$ **section** of P.*

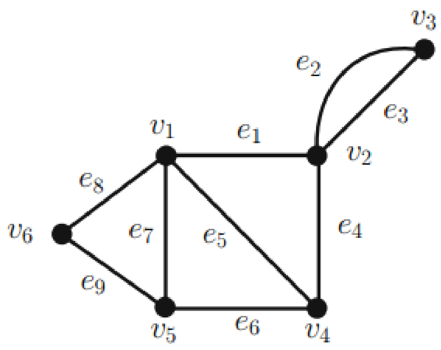


Figure 2.1: Graph illustrating walks,trails,paths and cycles

2.3 Bipartite graphs

Definition 2.5 *A graph is **bipartite** if its vertex set can be partitioned into two non-empty subsets X and Y such that each edge of G has one end in X and the other in Y. The pair (X, Y) is called a **bipartition** of the bipartite graph. The bipartite graph G with bipartition (X, Y) is denoted by $G(X, Y)$. A simple bipartite graph $G(X, Y)$ is **complete** if each vertex of X is adjacent to all the vertices of Y. If $G(X, Y)$ is complete with $|X| = p$ and $|Y| = q$, then $G(X, Y)$ is denoted by $K_{p,q}$. A complete bipartite graph of the form $K_{1,q}$ is called a **star**.*

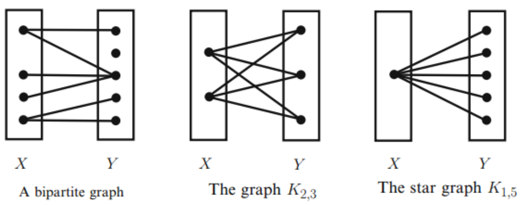


Figure 2.2: Bipartite graphs

Exercise 2.1 *(1) Give an example of a nonsimple disconnected graph with $\delta \geq \frac{n-1}{2}$.*

(2) Show that if G is a self-complementary graph of order n, then $n \equiv 0$ or 1 (mod 4).

(3) Show that if a self-complementary graph contains a pendant vertex, then it must have at least another pendant vertex.

(4) Prove that in a simple graph G, the union of two distinct paths joining two distinct vertices contains a cycle.

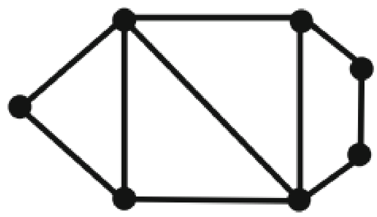
(5) *Show by means of an example that the union of two distinct walks joining two distinct vertices of a simple graph G need not contain a cycle.*

(6) *Prove or disprove: If H is a subgraph of G, then*

(a) $\delta(H) \leq \delta(G)$

(b) $\Delta(H) \leq \Delta(G)$.

(7) *In the following graph, find a closed trail of length 7 that is not a cycle:*



(8) *If $\delta \geq 2$, then show that G contains a cycle.*

Notes:

Chapter 3

TREES, CUT EDGES & VERTICES'S

Unit- III

3.1 Trees

Definition 3.1 *A connected acyclic graph is called a* ***tree****.*

Theorem 3.1 *A simple graph is a tree if, and only if, any two distinct vertices are connected by a unique path.*

Proof 3.1 *Let T be a tree. Suppose that two distinct vertices u and v are connected by two distinct $u - v$ paths. Then their union contains a cycle in T, contradicting that T is a tree.*
Conversely, suppose that any two vertices of a graph G are connected by a unique

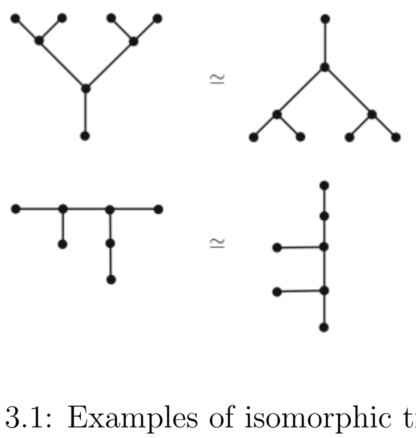


Figure 3.1: Examples of isomorphic trees

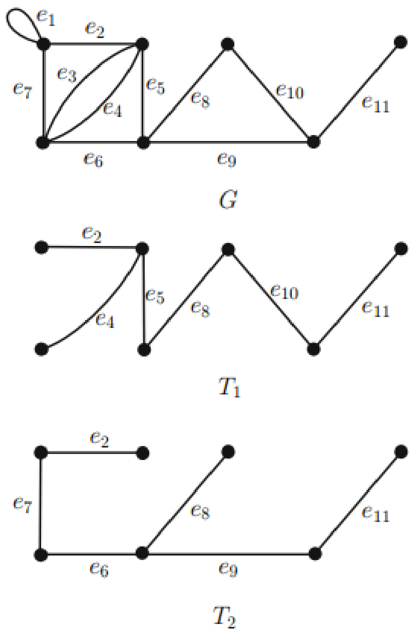


Figure 3.2: Graph G and two of its spanning trees

path. Then G is obviously connected. Also G can not contain a cycle, since any two distinct vertices of a cycle are connected by two distinct paths. Hence G is a tree.

Definition 3.2 *A spanning subgraph of a graph, which is also a tree, is called a **spannning tree** of the graph.*

Theorem 3.2 *Every connected graph contains a spanning tree.*

Theorem 3.3 *The number of edges in a tree with n vertices is $n-1$. Conversely, a connected graph with n vertices and $n-1$ edges is a tree.*

Theorem 3.4 *A tree with at least two vertices contains at least two pendant vertices.*

Proof 3.2 *Consider a longest path of a tree T. The end vertices of this path must be pendant vertices of T; otherwise, the path is extendable to a longer path or else T contains a cycle, a contradiction.*

Corollary 3.1 *If $\delta(G) \geq 2$, then G contains a cycle.*

Proof 3.3 *If G has no cycles, then G is a forest and hence $\delta(G) \leq 1$ by theorem* (7.1).

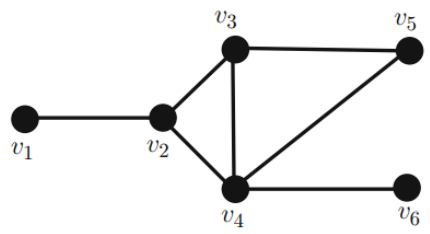


Figure 3.3: Graph illustrating vertex cuts and edge cuts

3.2 Cut edges and bonds

Definition 3.3 *Let G be a nontrivial connected graph with vertex set V and let S be a nonempty subset of V. For $\bar{S} = V\S$, let $[S, \bar{S}]$ denote the set of all edges of G that have one end vertex in S and tha other in $\bar{S}$. A set of edges of G of the form $[S, \bar{S}]$ is called an* **edge cut** *of G. An edge e is a* **cut edge** *of G, if $\{e\}$ is an edge cut of G.*

Example 3.1 *For the above graph, $\{v_2\}$ and $\{v_3, v_4\}$ are vertex cuts. The edge subsets $\{v_3v_5, v_4v_5\}, \{v_1v_2\}$ and $\{v_4v_6\}$ are all edge cuts. Of these, v_2 is a cut vertex, and $\{v_1v_2\}$ and $\{v_4v_6\}$ are both cut edges. For the edge cut $\{v_3v_5, v_4v_5\}$, we may take $S = \{v_5\}$ so that $\bar{S} = \{v_1, v_2, v_3, v_4, v_6\}$.*

Theorem 3.5 *An edge $e = xy$ of a graph G is a cut edge of a connected graph G if, and only if, e does not belong to any cycle of G.*

Proof 3.4 *Let e be a cut edge of G, and let $[S, \bar{S}] = \{e\}$ be the partition of V defined by $G - e$ so that $x \in S$ and $y \in \bar{S}$. If e belongs to a cycle of G, then $[S, \bar{S}]$ must contain at least one more edge, contradicting that $\{e\} = [S, \bar{S}]$. Hence e cannot belong toa cycle.*
Conversely, assume that e is not a cut edge of G. Then $G - e$ is connected, and hence there exists an $x - y$ path P in $G - e$. Then $P \cup \{e\}$ is a cycle in G containing e.

Theorem 3.6 *An edge $e = xy$ is a cut edge of a connected graph G if, and only if, there exist vertices u and v such that e belongs to every $u - v$ path in G.*

Proof 3.5 *Let $e = xy$ be a cut edge of G. Then $G - e$ has two components, G_1 and G_2. Let $u \in V(G_1)$ and $v \in V(G_2)$ Then clearly, every $u - v$ path in G contains e.*
Conversely, suppose that there exist vertices u and v satisfying the condition of the theorem. THen, there exists no $u - v$ path in $G - e$ so that $G - e$ is disconneccted. Hence e is a cut edge of G.

Proposition 3.1 *A simple cubic connected grasph G has a cut vertex if, and only if, it has a cut edge.*

3.3 Cut vertex

Definition 3.4 *A subset V' of the vertex set $V(G)$ of a connected graph G is a* ***vertex cut*** *of G, if $G - V'$ is disconnected; it is a $k - vertexcut$ if $|V'| = k$. V' is then called a* ***separating set*** *of vertices of G. A vertex v of G is a* ***cut vertex*** *of G, if $\{v\}$ is a vertex cut of G.*

Theorem 3.7 *A vertex v of a connected graph G with at least three vertices is a cut vertex of G if, and only if, there exist vertices u and w of G, distinct from v, such that v is in every $u - w$ path in G.*

Proof 3.6 *If v is a cut vertex of G, then $G - v$ is disconnected and has at least two components, G_1 and G_2. Take $u \in V(G_1)$ and $w \in V(G_2)$. THen every $u - w$ path in G must contain v, as otherwise u and w would belong to the same component of $G - v$.*
Conversely, suppose that the condition of the theorem holds. Then the deletion of v destroys every $u - w$ path in G, and hence u and w lie in distinct components of $G - v$. THerefore, $G - v$ is disconnected and v is a cut vertex of G.

3.4 Cayley's formula

Theorem 3.8 *The number of spanning trees of a complete labeled graph G on n vertices is $\tau(K_n) = n^{n-2}$ where $n \geq 2$.*

Before we prove Theorem (7.2), we establish two lemmas.

Lemma 3.9 *Let $(d_1, \ldots, d_n))$ be a sequence of positive integers with $\sum_{i=1}^{n} d_i = 2(n-1)$, then there exists a tree T with vertex set $\{v_1, \ldots, v_n\}$ and $d(v_i) = d_i$,$1 \leq i \leq n$.*

Proof 3.7 *It is easy to prove the result by induction on n.*

Lemma 3.10 *Let $\{v_1, \ldots, v_n\}$,$n \geq 2$ be given and let $\{d_1, \ldots, d_n\}$ be a set of positive integers such that $\sum_{i=1}^{n} d_i = 2(n - 1)$. Then the number of trees with $\{v_1, \ldots, v_n\}$ as the vertex set in which v_i has degree d_i,$1 \leq i \leq n$, is $\frac{(n-2)!}{(d_1-1)!\ldots(d_n-1)!}$.*

Proof 3.8 *We prove this result by induction on n.*

The total number of trees T_n with vertex set $\{v_1, \ldots, v_n\}$ is obtained by summing over all possible sequences $(d_1, \ldots, d_n)$ with $\sum_{i=1}^{n} d_i = 2n - 2$. Hence,

$$\tau(K_n) = \sum_{d_i \geq 1} \frac{(n-2)!}{(d_1 - 1)! \ldots (d_n - 1)!} \text{ with } \sum_{i=1}^{n} d_i = 2n - 2$$

$$= \sum_{k_i \geq 0} \frac{(n-2)!}{k_1! \ldots k_n!} \text{ with } \sum_{i=1}^{n} k_i = n - 2, \text{ where} k_i = d_i - 1, 1 \leq i \leq n$$

Putting $x_1 = x_2 = \ldots = x_n$ and $m = n-2$ in the multinomial expansion $(x_1 + \cdots + x_n)^m = \sum_{k_i \geq 0} \frac{x_1^{k_1} \ldots x_n^{k_n}}{k_1! \ldots k_n!} m!$ with $(k_1 + k_2 + \ldots + k_n) = m$, we get $n^{n-2} = \sum_{k_i \geq 0} \frac{(n-2)!}{k_1! \ldots k_n!}$ with $(k_1 + k_2 + \ldots + k_n) = n-2$. Thus $\tau(K_n) = n^{n-2}$.

Exercise 3.1 *(1) IF $\{x, y\}$ is a 2-edge cut of a graph G, show that every cycle of G that contains x must also contain y.*

(2) Prove or disprove: Let G be a simple connected graph with $n(G) \geq 3$. Then G has a cut edge iff G has a cur vertex.

(3) Show that in a graph, the number of edges common to a cycle and an edge cut is even.

(4) Give an example of a graph with n vertices and $n-1$ edges that is not a tree.

Notes:

Chapter 4

BLOCKS

Unit- IV

4.1 CONNECTIVITY

Definition 4.1 *For a nontrivial connected graph G having a pair of nonadjacent vertices, the minimum k for which there exists a k-vertex cut is called the* ***vertex connectivity*** *or simply the* ***connectivity*** *of G; it is denoted by $\kappa(G)$ or simply κ.*

Definition 4.2 *A set of vertices or edges of a connected graph G is said to* ***disconnect*** *the graph if its deletion results in a disconnected graph.*

Definition 4.3 *The* ***edge connectivity*** *of a connected graph G is the smallest k for which there exists a k-edge cut.*

Definition 4.4 *A graph G is r-**connected** if $\kappa(G) \geq r$. G is r-**edge connected** if $\lambda(G) \geq r$.*

Theorem 4.1 *For any loopless connected graph G, $\kappa(G) \leq \lambda(G) \leq \delta(G)$*

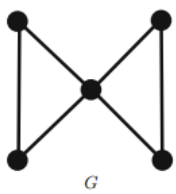


Figure 4.1: 1-connected graph

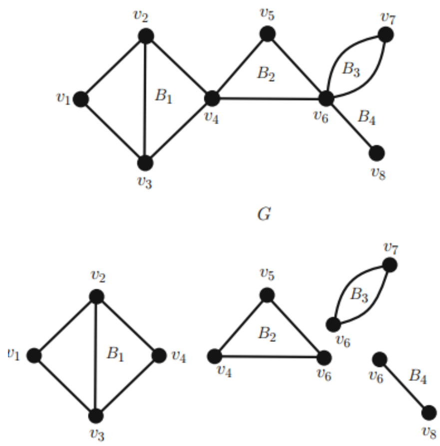


Figure 4.2: A graph G and its blocks

Theorem 4.2 *The connectivity and edge connectivity of a simple cubic graph G are equal.*

Definition 4.5 *A family of two or more paths in a graph G is said to be* ***internally disjoint*** *if no vertex of G i an internal vertex of more than one path in the family.*

Theorem 4.3 *A graph G with at least three vertices is 2-connected iff any two vertices of G are connected by at least two internally disjoint paths.*

4.2 Blocks

Definition 4.6 *A graph G is* ***nonseparable*** *if it si nontrivial, connected and has no cut vertices. A* ***block*** *of a graph G is a maximal nonseparable subgraph of G. If G has no cut vertex, G itself is a block.*

Theorem 4.4 *If C is any cycle of a simple block G with at least three vertices, then there exists a sequence of non-separable subgraphs* $c = B_0, B_1, \ldots, B_r = G$ *such that* B_{i+1} *is an edge-disjoint union of* B_i *and a path* P_i*,where the only vertices common to* B_i *and* P_i *are the end vertices of* P_i*,*$0 \leq i \leq r-1$.

Proof 4.1 *Assume that we already determined* B_i*. If* $B_i \neq G$*, there exists an edge* $e = uv$ *not belonging to* B_i*, but with u in* B_i*. If v also belongs to* B_i*, take* $P_i = uv$ *and* $B_{i+1} = B_i \cup P_i$*. Otherwise* $e = uv$ *is an edge of G having only one of its ends, namely u, in* B_i*. Let* u' *be any other vertex of* B_i*. Then , since G is 2-connected, e and* u' *belong to a common cycle* C_i*. Let* u_i *be the first vertex of* B_i *in the* $u - u'$ *section* C' *of* C_i *containing v, and let* P_i *be the* $u - u_i$ *section of* C'*. Define* $B_{i+1} = B_i \cup P_i$*. Then* B_{i+1} *is non-separable, and the proof follows by induction on i.*

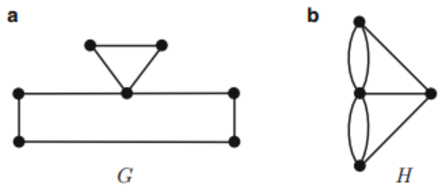


Figure 4.3: (a) Eulerian graph and (b) Non-Eulerian graph

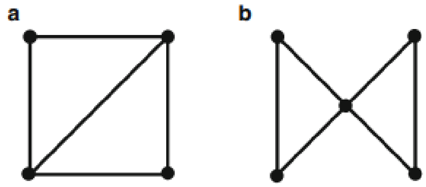


Figure 4.4: (a) Hamiltonian graph, (b) Non-Hamiltonian graph

4.3 Euler tours

Definition 4.7 *An* ***Euler trail*** *in a graph G is a spanning trail in G that contains all the edges of G. An* ***Euler tour*** *of G is a closed Eulertrail of G. G is called* ***Eulerian*** *if G has an Euler tour.*

Theorem 4.5 *For a connected graph G, the following statements are equivalent:*
(i) G is Eulerian
(ii) The degree of each vertex of G is an even positive integer.
(iii) G is an edge-disjoint union of cycles.

Theorem 4.6 *A graph is Eulerian if, and only if, each edge e of G belongs to an odd number of cycles of G.*

Corollary 4.1 *A graph is Eulerian if, and only if, it has an odd number of cycle decompositions*

4.4 Hamiltonion cycles

Definition 4.8 *A graph is called* ***Hamiltonian*** *if it has a spanning cycle. These are often called* ***Hamiltonian cycle*** *of G.*

Theorem 4.7 *If G is Hamiltonian, then for every non-empty proper subset S of V, $\omega(G-S) \leq |S|$.*

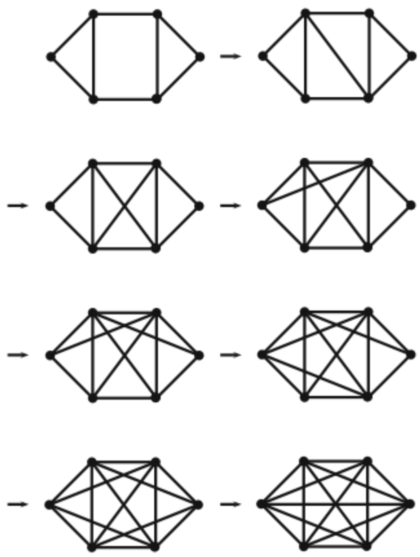


Figure 4.5: Closure of a graph

Proof 4.2 *Let C be a Hamiltonian cycle in G. Then, since C is a spanning subgraph of G, $\omega(G-S) \leq \omega(C-S)$. If $|S| = 1$, $C-S$ is a path, and therefore $\omega(C-S) = 1 = |S|$. The removal of a vertex from a path P results in one or two components, according to whether the removed vertex is an end vertex or an internal vertex of P, respectively. Hence, by induction, the number of components in $C-S$ cannot exceed $|S|$. This proves that $\omega(G-S) \leq \omega(C-S) \leq |S|$.*

Theorem 4.8 *Let G be a simple graph with $n \geq 3$ vertices. If for every pair of nonadjacent vertices u, v of G, $d(u) + d(v) \geq n$, then G is Hamiltonian.*

4.5 Closure of a graph

Definition 4.9 *The **closure** of a graph G, denoted by $cl(G)$ is defined to be the supergraph of G obtained from G by recursively joining pairs of nonadjacent vertices whose degree sum is at least n until no such pair exists.*

Theorem 4.9 *The closure $cl(G)$ of a graph G is well-defined.*

Theorem 4.10 *If $cl(G)$ is Hamiltonian, then G is Hamiltonian.*

Corollary 4.2 *If $cl(G)$ is complete, then G is Hamiltonian.*

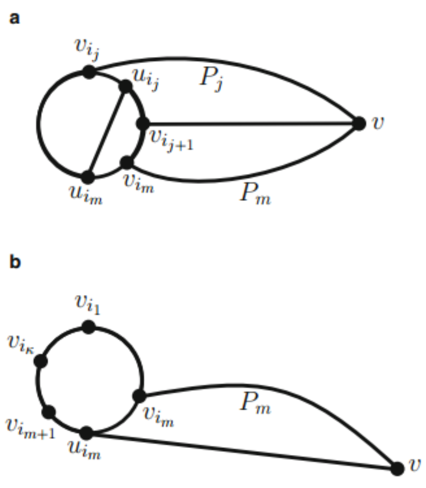


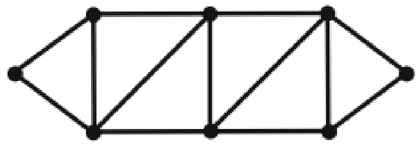
Figure 4.6: Graphs for proof of the theorem (4.11)

4.6 Chavatal theorem for non-Hamiltonian simple graphs

Theorem 4.11 *If for a simple 2-connected graph G, $\alpha \leq \kappa$, then G is Hamiltonian.*

Proof 4.3 *Suppose $\alpha \leq \kappa$ but G is not Hamiltonian. Let $C : v_0 v_1 \dots v_{p-1}$ be a longest cycle of G. We fix this orientation on C. By Dirac's theorem $p \geq \kappa$. Let $v \in V(G)\ V(C)$. Then by Menger's theorem there exist κ internally disjoint paths $P_1, \dots, P_\kappa$ from v to C. Let $v_{i_1}, v_{i_2}, \dots, v_{i_k}$ be the end vertices of the paths on C. No two of the consecutive vertices $v_{i_1}, v_{i_2}, \dots, v_{i_k}$, v_{i_1} can be adjacent vertices of C, since oterwise we get a cycle of G longer than C. Hence, between any two consecutive vertices of $\{v_{i_1}, v_{i_2}, \dots, v_{i_k}, v_{i_1}\}$, there exists at least one vertex of G. Let u_{i_j} be the vertex next to v_{i_j} in the $v_{i_j} - v_{i_j+1}$ path along C.*
We claim that $\{u_{i_1}, u_{i_2}, \dots, u_{i_k}\}$ is an independent set of G. Suppose u_{i_j} is adjacent to u_{i_m}, $m \geq j$; then $u_{i_j}, \dots v_{i_{j+1}} \dots v_{i_m} v v_{i_{j-1}} \dots u_{i_m} u_{i_j}$ is a cycle of G longer than C, a contradiction.
Clearly, $\{v, u_{i_1}, u_{i_2}, \dots, v_{i_k}\}$ is an independant set of G.(Otherwise, $vu_{i_m} \in E(G)$ for some m. Then $vu_{i_m} \dots \vDash_{i_{m+1}} \dots v_{i_k} \dots v_{i_1} \dots v_{i_m} P_m^{-1} v$ is a cycle longer than C, a contradiction) But than $\alpha \geq \kappa$, a contradiction to our assumption. Thus G is Hamiltonian.

Exercise 4.1 *(1) Determine the closure of the following graph.*
(2) Does there exist an Eulerian graph with (i) An even number of vertices and an odd number of edges? (ii) An odd number of vertices and an even number of edges? Draw such a graph if it exists.
(3) Show that in a tree, any path of maximum length contains the center of the tree.
(4) Show that a simple graph with ω components is a forest if and only if $m = n - \omega$.



(5) A vertex v of a tree T with at least three vertices is a cut vertex of T if and only if v is not a pendant vertex.
(6) Prove or disprove: If H is a subgraph of G; then $\kappa(H) \leq \kappa(G)$

Notes:

Chapter 5

PERFECT MATCHINGS

Unit- V

5.1 MATCHINGS

Definition 5.1 *A subset M of E is called a matching in G if its elements are links and no two are adjacent in G; the two ends of an edge in M are said to be matched under M. A matching M saturates a vertex v, and v is said to be $M-$ saturated, if some edge of M is incident with v; otherwise, v is $M-$ unsaturated. If every vertex of G is $M-$saturated, the matching M is perfect. M is a maximum matching if G has no matching M' with $|M'| > |M|$; clearly, every perfect matching is maximum.*

Let M be a matching in G. An $M-$alternating path in G is a path whose edges are alternately in E M and M. For example, the path $v_5v_8v_1v_7v_6$ in the graph of figure **??** *is an $M-$alternating path. An $M-$augmenting path is an $M-$alternating path whose origin and terminus are $M-$unsaturated.*

Theorem 5.1 *(Berge, 1957) A matching M in G is a maximum matching if and only if G contains no $M-$augmenting path.*

Proof: *Let M be a matching in G, and suppose that G contains an M-augmenting path $v_0v_1...v_{2m+1}$. Define $M' \subseteq E$ by*

$$M' = (M \setminus v_1v_2, v_3v_4, ..., v_{2m-1}) \cup \{v_0v_1, v_2v_3, ...v_{2m}v_{2m+1}\}$$

Then M' is a matching in G, and $|M'| = |M| + 1$. Thus M is not a maximum matching.
Conversely, suppose that M is not a maximum matching, and let M' be a maximum matching in G. Then

$$|M'| > |M| \tag{5.1}$$

Set $H = G[M \triangle M']$, where $M \triangle M'$ denotes the symmetric difference of M and M'.
Each vertex of H has degree either one or two in H, since it can be incident with at most one edge of M and one edge of M' Thus each component of H is either an even cycle with edges alternately in M and M', or else a path with edges alternately in M and M'. By equation 5.1, H contains more edges of M' than of M, and therefore some path component P of H must start and end with edges of M' The origin and terminus of P, being $M'-$saturated in H, are $M-$unsaturated in G. Thus P is an $M-$augmenting path in G.

Exercises
5.1.1 (a) Show that every $k-$cube has a perfect matching ($k \geq 2$).
(b) Find the number of different perfect matchings in k_{2n} and $k_{n,n}$.
5.1.2 Show that a tree has at most one perfect matching.
5.1.3 For each $k > 1$, find an example of a $k-$regular simple graph that has no perfect matching.

5.2 MATCHINGS AND COVERINGS IN BIPARTITE GRAPHS

For any set S of vertices in G, we define the neighbour set of S in G to be the set of all vertices adjacent to vertices in S; this set is denoted by $N_G(S)$. Suppose, now, that G is a bipartite graph with bipartition (X, Y). In many applications one wishes to find a matching of G that saturates every vertex in X. Necessary and sufficient conditions for the existence of such a matching were first given by Hall (1935).

Theorem 5.2 *Let G be a bipartite graph with bipartition (X, Y). Then G contains a matching that saturates every vertex in X if and only if*

$$|N(S)| \geq |S| \quad \textit{for all} \quad S \subseteq X \tag{5.2}$$

Proof:
Suppose that G contains a matching M which saturates every vertex in X, and let S be a subset of X. Since the vertices in S are matched under M with distinct vertices in $N(S)$, we clearly have $|N(S)| \geq |S|$.

Conversely, suppose that G is a bipartite graph satisfying equation 13.2, but that G contains no matching saturating all the vertices in X. We shall obtain a contradiction. Let $M*$ be a maximum matching in G. By our supposition, $M*$ does not saturate all vertices in X. Let u be an $M*-$unsaturated vertex in X, and let Z denote the set of all vertices connected to u by $M*-$ alternating paths. Since $M*$ is a maximum matching, it follows from theorem 1 that u is the only $M*-$unsaturated *vertex in Z. Set $S = Z \cap X$ and $T = Z \cap Y$. Clearly, the vertices in $S \setminus \{u\}$ are matched under $M*$ with the vertices in T. Therefore*

$$|T| = |S| - 1 \tag{5.3}$$

and $N(S) \supseteq T$. In fact, we have

$$N(S) = T \tag{5.4}$$

since every vertex in $N(S)$ is connected to u by an $M*-$alternating path. But equation 5.3 and 5.4 imply that $|N(S)| = |S| - 1 < |S|$ contradicting assumption 13.2.

Corollary 1:
If G is a $k-$regular bipartite graph with $k > 0$, then G has a perfect matching.

Exercises:
1. Show that it is impossible, using 1×2 rectangles, to exactly cover an 8×8 square from which two opposite 1×1 corner squares have been removed.

2. (A)Show that a bipartite graph G has a perfect matching if and only if $|N(S)| \geq |S|$ for all $S \subseteq V$.

5.3 PERFECT MATCHINGS

A necessary and sufficient condition for a graph to have a perfect matching was obtained by Tutte (1947). The proof given here is due to Lovasz (1973). A component of a graph is odd or even according as it has an odd or even number of vertices. We denote by $o(G)$ the number of odd components of G.

Theorem 5.3 *G has a perfect matching if and only if $o(G-S) \leq |S|$ for all $S \subset V$.*

Theorem 5.4 *Every 3-regular graph without cut edges has a perfect matching.*

Notes:

Chapter 6

INDEPENDENT SETS, CLIQUES & RAMSEY'S NUMBERS

Unit - VI

6.1 INDEPENDENT SETS & CLIQUES

A subset S of V is called an independent set of G if no two vertices of S are adjacent in G. An independent set is maximum if G has no independent set S' with $|S'| > |S|$. Recall that a subset K of V such that every edge of G has at least one end in K is called a covering of G.

Theorem 6.1 *A set S is an independent set of G if and only if $V \setminus S$ is a covering of G.*
Proof:
By definition, S is an independent set of G if and only if no edge of G has both ends in S or, equivalently, if and only if each edge has at least one end in $V \setminus S$. But this is so if and only if $V \setminus S$ is a covering of G

The number of vertices in a maximum independent set of G is called the independence number of G and is denoted by $\alpha(G)$; similarly, the number of vertices in a minimum covering of G is the covering number of G and is denoted by $\beta(G)$.

Theorem 6.2 $\alpha + \beta = \nu$.
Proof:
Let S be a maximum independent set of G, and let K be a minimum covering of G. Then, by theorem 7.1, $V \setminus K$ is an independent set and $V \setminus S$ is a covering. Therefore,

$$\nu - \beta = |V \setminus K| \leq \alpha \tag{6.1}$$

$$\nu - \alpha = |V \setminus S| \geq \beta \tag{6.2}$$

combining equation 6.1 and 6.2, we have $\alpha + \beta = \nu$.

The edge analogue of an independent set is a set of links no two of which are adjacent, that is, a matching. The edge analogue of a covering is called an edge covering. An edge covering of G is a subset L of E such that each vertex of G is an end of some edge in L. Note that edge coverings do not always exist; a graph G has an edge covering if and only if $\delta > 0$. We denote the number of edges in a maximum matching of G by $\alpha'(G)$, and the number of edges in a minimum edge covering of G by $\beta'(G)$; the numbers $\alpha'(G)$ and $\beta'(G)$ are the edge independence number and edge covering number of G, respectively.

Matchings and edge coverings are not related to one another as simply as are independent sets and coverings; the complement of a matching need not be an edge covering, nor is the complement of an edge covering necessarily a matching. However, it so happens that the parameters α' and β' are related in precisely the same manner as are α and β.

Theorem 6.3 *(Gallai, 1959) If* $\delta > 0$, *then* $\alpha' + \beta' = \nu$.
Proof:
Let M *be a maximum matching in* G *and let* U *be the set of* $M-$*unsaturated vertices. Since* $\delta > 0$ *and* M *is maximum, there exists a set* E' *of* $|U|$ *edges, one incident with each vertex in* U. *Clearly,* $M \cup E'$ *is an edge covering of* G, *and so*

$$\alpha' + \beta' \leq \nu \tag{6.3}$$

Now let L *be a minimum edge covering of* G, *set* $H = G[L]$ *and let* M *be a maximum matching in* H. *Denote the set of* $M-$*unsaturated vertices in* H *by* U. *Since* M *is maximum,* $H[U]$ *has no links and therefore*

$$|L| - |M| = |L \setminus M| \geq |U| = \nu - 2|M|$$

Because H *is a subgraph of* G, M *is a matching in* G *and so*

$$\alpha' + \beta' \geq |M| + |L| \geq \nu \tag{6.4}$$

Combining equation 6.3 and 6.4, we have $\alpha' + \beta' = \nu$

Theorem 6.4 *In a bipartite graph* G *with* $\delta > 0$, *the number of vertices in a maximum independent set is equal to the number of edges in a minimum edge covering.*

6.2 RAMSEY'S THEOREM

In this section we deal only with simple graphs. A clique of a simple graph G is a subset S of V such that $G[S]$ is complete. Clearly, S is a clique of G if and only if

S is an independent set of G^c, and so the two concepts are complementary.
If G has no large cliques, then one might expect G to have a large independent set. That this is indeed the case was first proved by Ramsey (1930). He showed that, given any positive integers k and l , there exists a smallest integer $r(k,l)$ such that every graph on $r(k,l)$ vertices contains either a clique of k vertices or an independent set of l vertices. For example, it is easy to see that

$$r(1,l) = r(k,1) = 1 \tag{6.5}$$

and

$$r(2,l) = l, r(k,2) = k \tag{6.6}$$

The numbers $r(k,l)$ are known as the Ramsey numbers.

Theorem 6.5 *For any two integers $k \geq 2$ and $l \geq 2$*

$$r(k,l) \leq r(k,l-1) + r(k-1,l) \tag{6.7}$$

Furthermore, if $r(k,l-1)$ and $r(k-1,l)$ are both even, then strict inequality holds in equation 6.7.
Proof:

Proof Let G be a graph on $r(k,l-1) + r(k-1,l)$ vertices, and let $v \in V$. We distinguish two cases:
(i) v is nonadjacent to a set S of at least $r(k,l-1)$ vertices, or
(ii) v is adjacent to a set T of at least $r(k-1,l)$ vertices.
Note that either case (i) or case (ii) must hold because the number of vertices to which v is nonadjacent plus the number of vertices to which t; is adjacent is equal to $r(k,l-1) + r(k-1,l) - 1$.

In case (i), $G[S]$ contains either a clique of k vertices or an independent set of $l-1$ vertices, and therefore $G[S \cup \{v\}]$ contains either a clique of k vertices or an independent set of l vertices. Similarly, in case (ii), $G[T \cup \{v\}]$ contains either a clique of k vertices or an independent set of l vertices. Since one of case (i) and case (ii) must hold, it follows that G contains either a clique of k vertices or an independent set of l vertices. This proves equation 6.7.

Now suppose that $r(k,l-1)$ and $r(k?1,l)$ are both even, and let G be a graph on $r(k,l?1) + r(k?1,l) - 1$ vertices. Since G has an odd number of vertices, it follows from corollary 1 that some vertex v is of even degree; in particular, v cannot be adjacent to precisely $r(k-l,l)-1$ vertices. Consequently, either case (i) or case (ii) above holds, and therefore G contains either a clique of k vertices or an independent set of l vertices. Thus, $r(k,l) \leq r(k,l-1) + r(k-1,l) - 1$ as stated.

Theorem 6.6 $r(k,k) > 2^{k/2}$

Theorem 6.7 *If* $m = mink, l$ *then* $r(k, l) \geq 2^{m/2}$

Notes:

Chapter 7

EDGE COLOURINGS

Unit - VII

7.1 EDGE CHROMATIC NUMBER

A k-edge colouring $\mathcal{C}$ of a loopless graph G is an assignment of k colours, $1, 2, \ldots, k$, to the edges of G. The colouring $\mathcal{C}$ is proper if no two adjacent edges have the same colour. Alternatively, a k-edge colouring can be thought of as a partition $(E_1, E_2, \ldots, E_k)$ of E, where B denotes the (possibly empty) subset of E assigned colour i. A proper k-edge colouring is then a k-edge colouring $(E_1, E_2, \ldots, E_k)$ in which each subset E_i is a matching.

G is k-edge colourable if G has a proper k-edge-colouring- Trivially, every loopless graph G is ϵ-edge-colourable; and if G is k-edge-colourable, then G is also l-edge-colourable for every $l > k$. The edge chromatic number $\chi'(G)$, of a loopless graph G, is the minimum k for which G is k-edge- colourable. G is k-edge-chromatic if $\chi'(G) = k$. It can be readily verified that the graph of figure 6.1 has no proper 3-edge colouring. This graph is therefore 4-edge-chromatic. Clearly, in any proper edge colouring, the edges incident with any one vertex must be assigned different colours. It follows that

$$\chi' \geq k \tag{7.1}$$

Lemma 7.1 *Let G be a connected graph that is not an odd cycle. Then G has a 2-edge colouring in which both colours are represented at each vertex of degree at least two.*

Proof 7.1 *We may clearly assume that G is nontrivial. Suppose, first, that G is*

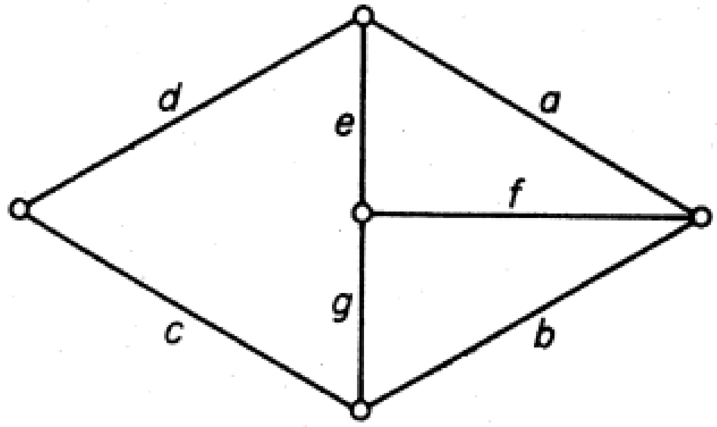


Figure 7.1: 1

eulerian. If G is an even cycle, the proper 2-edge colouring of G has the required property. Otherwise, G has a vertex v_0 *of degree at least four. Let be an Euler tour of G, and set* $v_0 e_1 v_1 \ldots e_\epsilon v_0$

$$E_1 = \{e_i | i \ odd\} \text{and} E_2 = \{e_i | i \ even\} \tag{7.2}$$

Then the 2-edge colouring (E_1, E_2) *of G has the required property, since each vertex of G is an internal vertex of* $v_0 e_1 v_1 \ldots e_\epsilon v_0$*. If G is not eulerian, construct a new graph* G^* *by adding a new vertex v0 and joining it to each vertex of odd degree in G. Clearly* G^* *is eulerian. Let* $v_0 e_1 v_1 \ldots e_\epsilon v_0$ *be an Euler tour of* G^* *and define* E_1 *and* E_2 *as in (7.1). It is then easily verified that the 2-edge colouring* $(E_1 \cap E, E_2 \cap E)$ *of G has the required property.*

Lemma 7.2 *Let* $\mathcal{C} = (E_1, E_2, \ldots, E_k)$ *be an optimal k-edge colouring of G. If there is a vertex u in G and colours i and j such that i is not represented at u and j is represented at least twice at u, then the component of* $G[E_i \cup E_j]$ *that contains u is an odd cycle.*

Proof 7.2 *Let u be a vertex that satisfies the hypothesis of the lemma, and denote by H the component of* $G[E_i \cup E_j]$ *containing u. Suppose that H is not an odd cycle. Then, by lemma 7.1 H has a 2-edge colouring in which both colours are represented at each vertex of degree at least two in H. When we recolour the edges of H with colours i and j in this way, we obtain a new k-edge colouring* $\mathcal{C}' = (E_1', E_2', \ldots, E_k')$ *of G. Denoting by* $c'(v)$ *the number of distinct colours at v in the colouring* $\mathcal{C}'$*, we have*

$c'(u) = c(u) + 1$ since, now, both i and j are represented at u, and also $c'(v) \geq c(v)$ for $u \neq v$. Thus $\sum_{v \in V} c'(v) > \sum_{v \in V} c(v)$, contradicting the choice of $\mathcal{C}'$. It follows that H is indeed an odd cycle .

Theorem 7.1 *If G is bipartite, then $\chi' = \Delta + 1$.*

Proof 7.1 *Let G be a graph with $\chi' > \Delta + 1$, let $\mathcal{C}' = (E'_1, E'_2, \ldots, E'_\Delta)$ be an optimal A-edge colouring of G, and let u be a vertex such that $c(u) < d(u)$. Clearly, u satisfies the hypothesis of lemma 7.2. Therefore G contains an odd cycle and so is not bipartite. It follows from* (7.1) *that if G is bipartite, then $\chi' = \Delta + 1$.*

Exercises:

1. Show that the Petersen graph is 4-edge-chromatic.
2. Describe a good algorithm for finding a proper A-edge colouring of a bipartite graph G.

7.2 VIZING'S THEOREM

As has already been noted, if G is not bipartite then we cannot necessarily conclude that $\chi' = \Delta$. An important theorem due to Vizing A964) and, independently, Gupta A966), asserts that, for any simple graph G, either $\chi' = \Delta$ or $\chi' = \Delta + 1$. The proof given here is by Fournier 1973).

Theorem 7.2 *If G is simple, then either $\chi' = \Delta$ or $\chi' = \Delta + 1$.*

Proof 7.2 *Let G be a simple graph. By virtue of* (7.1) *we need only show that $\chi' \leq \Delta + 1$ Suppose, then, that $\chi' > \Delta + 1$. Let $\mathcal{C} = (E_1, E_2, \ldots, E_\Delta)$ be an optimal $(\Delta + 1)$-edge colouring of G and let u be a vertex such that $c(u) < d(u)$. Then there exist colours i_0 and i_1 such that i_0 is not represented at u, and i_1 is represented at least twice at u. Let uvi have colour i_1, as in figure 7.2a Since $d(v_1) < \Delta + 1$, some colour i_2 is not represented at v_1. Now i_2 must be represented at u since otherwise, by recolouring uvx with i_2, we would obtain an improvement on $\mathcal{C}$. Thus some edge uv_2 has colour i_2. Again, since $d(v_2) < \Delta + 1$, some colour i_3 is not represented at*

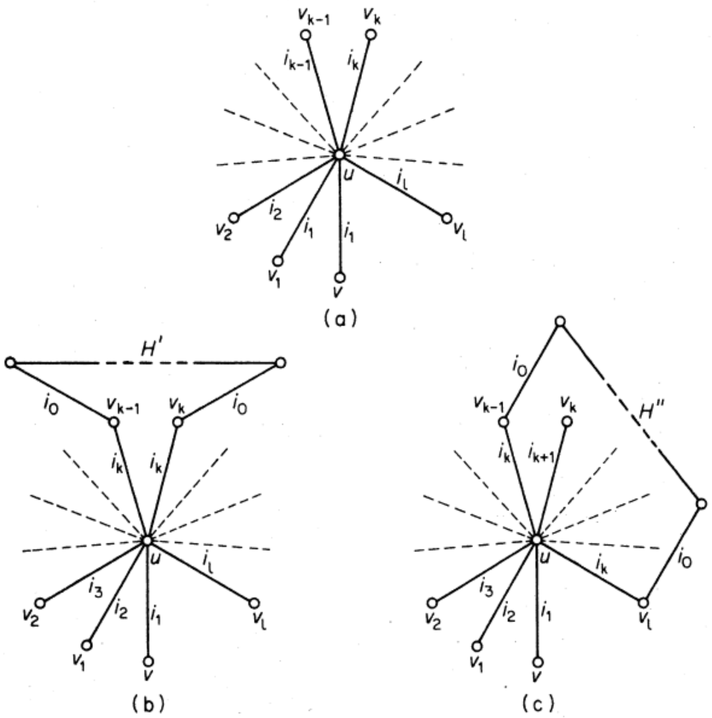


Figure 7.2: 2

v_2; and i_3 must be represented at u since otherwise, by recolouring uv_1 with i_2 and uv_2 with i_3, we would obtain an improved $(\Delta+1)$-edge colouring. Thus some edge uv_3 has colour i_3. Continuing this procedure we construct a sequence $i_1, i_2, \ldots$ of vertices and a sequence $i_1, i_2, \ldots$ of colours, such that

(i) *uv_j has colour i_j*

(ii) *i_{j+1} is not represented at v_j*
Since the degree of u is finite, there exists a smallest integer l such that, for some $k < l$,

(iii) *$i_{l+1} = i_k$.*

The situation is depicted in figure 7.2a. We now recolour G as follows. For $1 \leq j \leq k-1$ recolour $uv_j$with colour i_{j+i}, yielding a new $(\Delta+1)$-edge colouring $\chi' > \Delta+1$, let $\mathcal{C}' = (E'_1, E'_2, \ldots, E'_{\Delta+1})$. (7.2b) Clearly $c'(v) \geq c(v)$ for all $v \in V$ and therefore $\mathcal{C}'$ is also an optimal $(\Delta+1)$-edge colouring of G. By lemma 7.2, the component H' of $G[E'_{i_0} \cup E'_{i_k}]$ that contains u is an odd cycle. Now, in addition, recolour uv_j with colour i_{j+i}, $k \leq j \leq l?1$, and uv_l with colour i_k, to obtain a $(\Delta+1)$-edge colouring $\mathcal{C}' = (E''_1, E''_2, \ldots, E''_{\Delta+1})$ (7.2c). As above $c''(v) \geq c(v)$ for all $v \in V$ and *the component H'' of $G[E''_{i_0} \cup E''_{i_k}]$ that contains u is an odd cycle. But, since v_k has degree two in H', u_k clearly has degree one in H'' This contradiction establishes the theorem.*

Notes:

Exercises:

1. *Show that if G is loopless, then G has a A-regular loopless supergraph.*
2. *G is called uniquely k-edge-colourable if any two proper k-edge colourings of G induce the same partition of E. Show that every uniquely 3-edge-colourable 3-regular graph is hamiltonian.*

Chapter 8

VERTEX COLORING

Unit - VIII

8.1 CHROMATIC NUMBER

A k-vertex colouring of G is an assignment of k colours, $1, 2, \ldots, k$, to the vertices of G; the colouring is proper if no two distinct adjacent vertices have the same colour. Thus a proper k-vertex colouring of a loopless graph G is a partition $(V_1, V_2, \ldots, V_k)$ of V into k (possibly empty) independent sets. G is k-vertex-colourable if G has a proper k-vertex colouring. It will be convenient to refer to a 'proper vertex colouring' as, simply, a colouring and to a 'proper k-vertex colouring' as a k-colouring; we shall similarly abbreviate 'k-vertex-colourable' to k-colourable. Clearly, a graph is k-colourable if and only if its underlying simple graph is k-colourable. Therefore, in discussing colourings, we shall restrict ourselves to simple graphs; a simple graph is 1-colourable if and only if it is empty, and 2-colourable if and only if it is bipartite. The chromatic number, $\chi(G)$, of G is the minimum k for which G is k-colourable; if $\chi(G) = k$, G is said to be k-chromatic. A 3-chromatic graph is shown in figure 8.1. It has the indicated 3-colouring, and is not 2-colourable since it is not bipartite. It is helpful, when dealing with colourings, to study the properties of a special class of graphs called critical graphs. We say that a graph G is critical if $\chi(H) < \chi(G)$ for every proper subgraph H of G. Such graphs were first investigated by Dirac A952). A k-critical graph is one that is k-chromatic and critical; every k-chromatic graph has a k-critical subgraph. A 4-critical graph, due to Grotzsch A958), is shown in figure 8.2. An easy consequence of the definition is that every critical graph is connected. The following theorems establish some of the basic properties of critical graphs.

Theorem 8.1 *If G is k -critical, then $\delta \geq k-1$*

Proof 8.1 *By contradiction. If possible, let G be a k-critical graph with $\delta < k-1$, and let v be a vertex of degree δ in G. Since G is k-critical, $G-v$ is $(k-1)$-colourable. Let $(V_1, V_2, \ldots, V_{k-1})$ be a $(k-1)$-colouring of $G-v$. By definition, v is adjacent in G to $\delta < k-1$ vertices, and therefore v must be nonadjacent in G to every vertex of some V_j. But then $(V_1, V_2, \ldots, V_j \cup v, \ldots, V_{k-1})$ is a $(k-l)$-colouring of G, a contradiction. Thus $\delta < k-1$*

Theorem 8.2 *In a critical graph, no vertex cut is a clique.*

Proof 8.2 *By contradiction. Let G be a k-critical graph, and suppose that G has a vertex cut S that is a clique. Denote the S -components of G $G_1, G2, \ldots, G_n$. Since G is k-critical, each Gi is $(k-1)$-colourable. Furthermore, because S is a clique, the vertices in S must receive distinct colours in any $(k-1)$-colouring of Gi. It follows that there are $(k-1)$-colourings of $G_1, G_2, \ldots, G_n$ which agree on S. But these colourings together yield a $(k-1)$-colouring of G, a contradiction.*

Corollary 8.1 *Every critical graph is a block.*

Proof 8.1 *If $t¿$ is a cut vertex, then v is a vertex cut which is also, trivially, a clique. It follows from theorem 8.2 that no critical graph has a cut vertex; equivalently, every critical graph is a block*

Corollary 8.2 *Let G be a k-critical graph with a 2-vertex cut $\{u, v\}$. Then*

1. *$G = G_1 \cup G_2$, where G_1 is a $\{u, v\}$-component of type i $(i = 1, 2)$, and*
2. *both $G_1 + uv$ and $G_2 \cdot uv$ are k-critical .*

8.2 Brooks' theorem

Theorem 8.3 *If G is a connected simple graph and is neither an odd cycle nor a complete graph, then $\chi \leq \Delta$.*

Proof 8.2 *Let G be a k-chromatic graph which satisfies the hypothesis of the theorem. Without loss of generality, we may assume that G is k-critical. By corollary 8.1, G is a block. Also, since 1-critical and 2-critical graphs are complete and 3-critical graphs are odd cycles, we have $k \geq 4$ If G has a 2-vertex cut $\{u, v\}$, corollary8.2 gives $2\Delta \geq d(u) + d(c) \geq 3k - 5 \geq 2k - 1$. This implies that $\chi = k \leq \Delta$, since 2Δ is even.*

Assume, then, that G is 3-connected. Since G is not complete, there are three vertices u, v and w in G such that uv, $vw \in E$ and $uw \notin E$. Set $u = V_1$ and $w = v_2$ let $\{v_3, v_4, \ldots v_v = v\}$be any ordering of the vertices of $G - \{u, w\}$ such that each v_i is adjacent to some v_j with $j > i$. Finally, since v_v is adjacent to two vertices of colour 1 (namely t)i and D2), it is adjacent to at most $\Delta - 2$ other colours and can be assigned one of the colours $2, 3, \ldots, \Delta$,

8.3 Hajo's' Conjecture

A subdivision of a graph G is a graph that can be obtained from G by a sequence of edge subdivisions. A subdivision of K_4 is shown in figure 8.5. Although no necessary and sufficient condition for a graph to be k- chromatic is known when $k \geq 3$, a plausible necessary condition has been proposed by Hajos A961): if G is k -chromatic, then G contains a subdivi- sion of K_k. This is known as Hajos' conjecture. It should be noted that the condition is not sufficient; for example, a 4-cycle is a subdivision of K_3, but is not 3-chromatic.

For $k = 1$ and $k = 2$, the validity of Hajos' conjecture is obvious. It is also easily verified for $k = 3$, because a 3-chromatic graph necessarily contains an odd cycle, and every odd cycle is a subdivision of K3. Dirac A952) settled the case $k = 4$.

Theorem 8.4 *If G is 4-chromatic, then G contains a subdivision of K_4.*

Proof 8.3 *Let G be a 4-chromatic graph. Note that if some subgraph of G contains a subdivision of K_4, then so, too, does G. Without loss of generality, therefore, we may assume that G is critical, and hence that G is a block with $\Delta \geq 3$. If $v = 4$, then G is K_4 and the theorem holds trivially. We proceed by induction on v.*

Assume the theorem true for all 4-chromatic graphs with fewer than n vertices, and let $v(G) = n > 4$. Suppose, first, that G has a 2-vertex cut $\{u, v\}$. By theorem 8.3, G has two $\{u, v\}$-components G_1 and G_2, where $G_1 + uv$ is 4-critical. Since

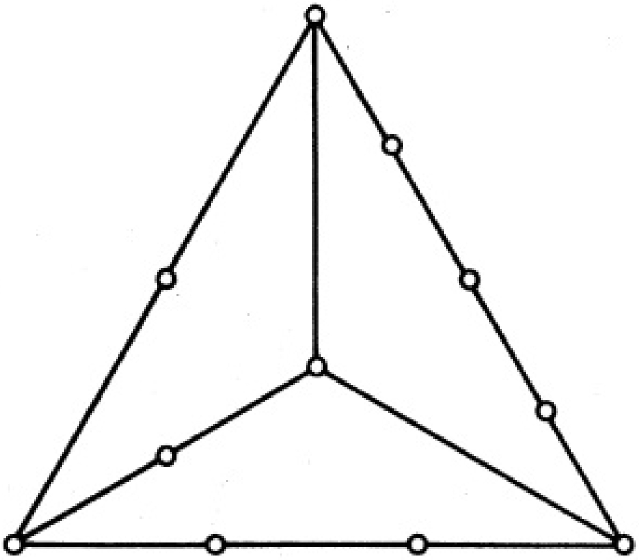


Figure 8.1: Subdivision of K_4

$v(G_1 + uv) < v(G)$, we can apply the induction hypothesis and deduce that $G_1 + uv$ contains a subdivision of K_4. It follows that, if P is a (u, v)-path in G_2, then $G \cup P$ contains a subdivision of K_4. Hence so, too, does G, since $G_1 \cup P \subset G$. Now suppose that G is 3-connected. Since $\Delta \geq 3$, G has a cycle C of length at least four. Let u and v be nonconsecutive vertices on C. Since $G - \{u, v\}$ is connected, there is a path P in $G - \{u, v\}$ connecting the two components of $C - \{u, v\}$ we may assume that the origin x and the terminus y are the only vertices of P on C. Similarly, there is a path Q in $G - \{x, y\}$. If P and Q have no vertex in common, then $C \cup P \cup Q$ is a subdivision of K_4. Otherwise, let w be the first vertex of P on Q, and let P' denote the (x, w)-section of P. Then $C \cup P' \cup Q$ is a subdivision of K_4 (figure 8.1). Hence, in both cases, G contains a subdivision of K_4

8.4 CHROMATIC POLYNOMIALS

In the study of colourings, some insight can be gained by considering not only the existence of colourings but the number of such colourings; this approach was developed by Birkhoff 1912) as a possible means of attacking the four-colour conjecture.

Theorem 8.5 *If G is simple, then $\pi_k(G) = \pi_k(G - e) - \pi_k(G \cdot e)$ for any edge e of G.*

Exercises:

- If a k-chromatic graph G has a colouring in which each colour is assigned to at least two vertices, show that G has a k-colouring of this type.
- Use Brooks' theorem to show that if G is loopless with $\Delta = 3$, then $\chi' \leq 4$.

Notes:

Chapter 9

PLANER GRAPHS

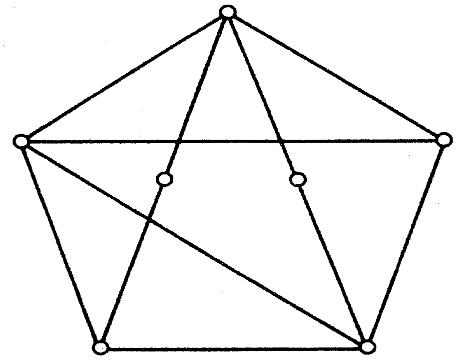
Unit - IX

9.1 PLANE AND PLANAR GRAPHS

A graph is said to be embeddable in the plane, or planar, if it can be drawn in the plane so that its edges intersect only at their ends. Such a drawing of a planar graph G is called a planar embedding of G. A planar embedding G of G can itself be regarded as a graph isomorphic to G; the vertex set of G is the set of points representing vertices of G, the edge set of G is the set of lines representing edges of G, and a vertex of G is incident with all the edges of G that contain it. We therefore sometimes refer to a planar embedding of a planar graph as a plane graph. Figure 2 (b) shows a planar embedding of the planar graph in figure 1 (a).

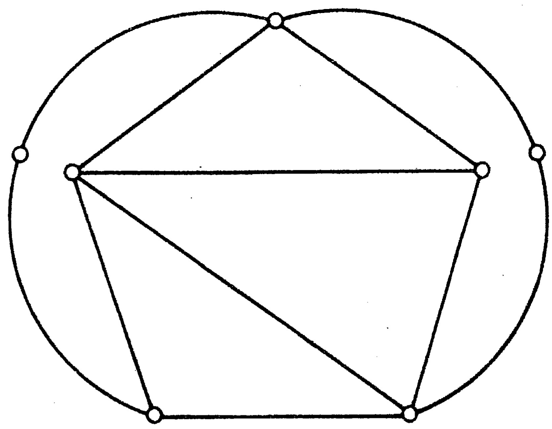
It is clear from the above definition that the study of planar graphs necessarily involves the topology of the plane. However, we shall not a tempt here to be strictly rigorous in topological matters, and will be content to adopt a naive point of view toward them. This is done so as not to obscure the combinatorial aspect of the theory, which is our main interest.

The results of topology that are especially relevant in the study of planar graphs are those which deal with Jordan curves. (A Jordan curve is a continuous non-self-intersecting curve whose origin and terminus coincide.) The union of the edges in a cycle of a plane graph constitutes a Jordan curve; this is the reason why properties of Jordan curves come into play in planar graph theory. We shall recall a well-known theorem about Jordan curves and use it to demonstrate the nonplanarity of K_5.



(a)

Figure 9.1: A planar graph G



(b)

Figure 9.2: A planar embedding of G

Let J be a Jordan curve in the plane. Then the rest of the plane is partitioned into two disjoint open sets called the interior and exterior of J. We shall denote the interior and exterior of J, respectively, by int J and ext J, and their closures by Int J and Ext J. Clearly Int J n Ext J = J. The Jordan curve theorem states that any line joining a point in int J to a point in ext J must meet J in some point (see figure 3). Although this theorem is intuitively obvious, a formal proof of it is quite difficult.

Theorem 9.1 *Ks is nonplanar.*

Proof 9.1 *By contradiction. If possible let G be a plane graph corresponding to* K_5*. Denote the vertices of G by* v_1, v_2, v_3, v_4 *and* v_5*. Since G is complete, any two of its vertices are joined by an edge. Now the cycle* $C = v_l v_2 v_3 v_1$ *is a Jordan curve in the*

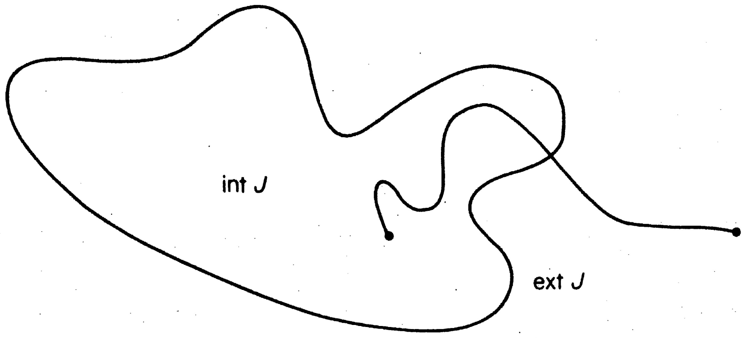


Figure 9.3:

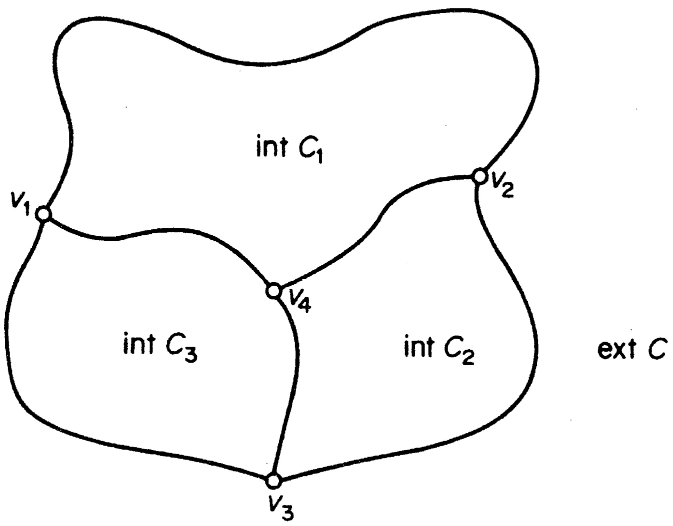


Figure 9.4:

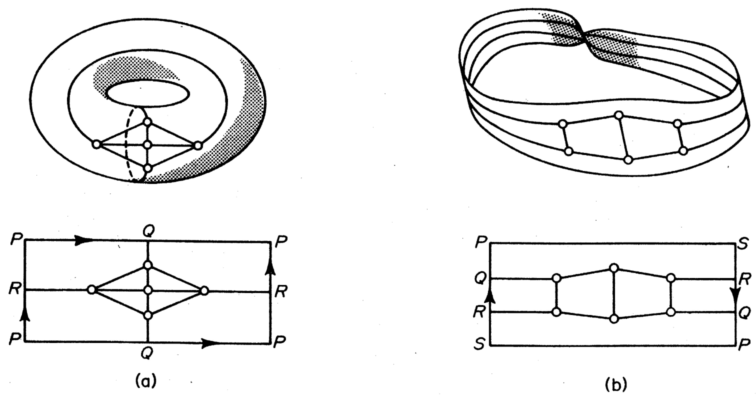


Figure 9.5: (a)An embedding of K_5 on the torus; (b) An embedding of $K_{3,3}$ on the Möbius band

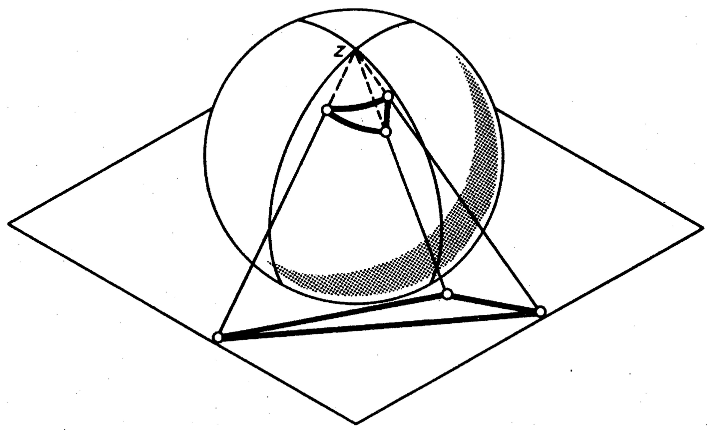


Figure 9.6: Stereographic Projection

plane, and the point v_4 must lie either in int C or ext C. We shall suppose that $v_4 \in$ int C. (The case where $v_4 \in$ ext C can be dealt with in a similar manner.) Then the edges v_4v_l, v_4v_2 and v_4v_3 divide int C into the three regions int C_1, int C_2 and int C_3, where $C_1 = v_1v_4v_2v_1, C_2 = v_2v_4v_3v_2$ and $C_3 = v_3v_4v_1v_3$ (see figure 4).

Now v_5 must lie in one of the four regions ext C, int C_1 , int C_2 and int C_3. If $v_5 \in$ ext C then, since $v_4 \in$ int C, it follows from the Jordan curve theorem that' the edge v_4v_5 must meet C in some point. But this contradicts the assumption that G is a plane graph. The cases $v_5 \in$ int C_i, i = 1, 2, 3, can be disposed of in like manner.

A similar argument. can be used to establish that $K_{3,3}$, too, is nonplanar. We shall see in section 9.5 that, on the other hand, every nonplanar graph contains a subdivision of either K_5 or $K_{3,3}$.

The notion of a planar embedding extends to other, surfaces. A graph G is said to be embeddable on a surface S if it can be drawn in S so that its edges intersect only at their ends; such a drawing (if one exists) is called an embedding of G on S. Figure 5 (a) shows an embedding of K_5 on the torus, and figure 5 (b) an embedding of $K_{3,3}$ on the Mobius band. The torus is represented as a rectangle in which opposite sides are identified, and the Mobius band as a rectangle whose two ends are identified after one half-twist.

We have seen that not all graphs can be embedded in the plane; this is also true of other surfaces. It can be shown (see, for example, Freshet and Fan, 1967) that, for every surface S, there exist graphs which are not embeddable on S. Every graph can, however, be 'embedded' in 3- dimensional space $\mathcal{R}^3$ (exercise 9.1.3).

Planar graphs and graphs embeddable on the sphere are one and the same. To show this we make use of a mapping known as stereographic projection. Consider a sphere S resting on a plane P, and denote by z the point of S that is diagonally opposite the point of contact of Sand P. The mapping π : S / {z} $\rightarrow$ P, defined by $\pi(S) = p$ if and only if the points z, s and p are collinear, is called stereographic projection from z; it is illustrated in figure 9.5.

Theorem 9.2 *A graph G is embeddable in the plane if and only if it is embeddable on the sphere.*

Proof 9.2 *Suppose G has an embedding $\tilde{G}$ on the sphere. Choose a point z of the sphere not in $\tilde{G}$. Then the image of $\tilde{G}$ under stereographic projection from z is an embedding of G in the plane. The converse is proved similarly.*

On many occasions it is advantageous to consider embeddings of planar graphs on the sphere; one instance is provided by the proof of theorem 9.3 in the next section.

Exercises

1. Show that $K_{3,3}$ is nonplanar.·
2. (a) Show that K_5 - e is planar for any edge e of K_5.
   (b) Show that $K_{3,3}$ - e is planar for any edge e of $K_{3,3}$.

9.2 DUAL GRAPHS

A plane graph G partitions the rest of the plane into a number of connected regions; the closures of these regions are called the faces of G. Figure 7 shows a plane graph with six faces, f_1, f_2, f_3, f_4, f_5 *and* f_6. The notion of a face applies also to embeddings of graphs on other surfaces. We shall denote by F(G) and $\phi(G)$, respectively, the set of faces and the number of faces of a plane graph G.

Each plane graph has exactly one unbounded face, called the exterior face; in the plane graph of figure 7, f_1 is the exterior face.

Theorem 9.3 *Let v be a vertex of a planar graph G. Then G can be embedded in the plane in such a way that v is on the exterior face of the embedding.*

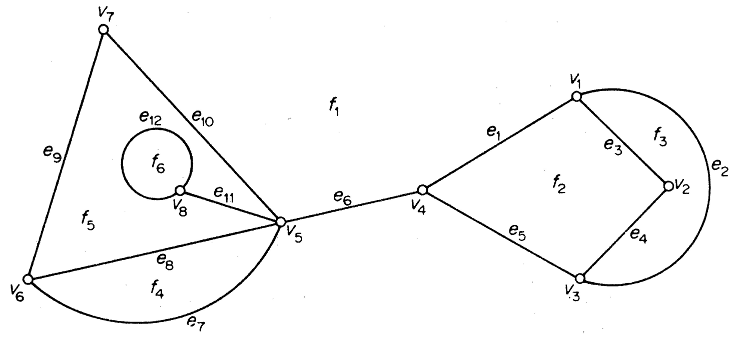


Figure 9.7: A plane graph with six faces

Proof 9.3 *Consider an embedding $\tilde{G}$ of G on the sphere; such an embedding exists by virtue of theorem 9.2. Let z b.e a point in the interior of some face containing v, and let $\pi(\tilde{G})$ be the image of $\tilde{G}$ under stereographic projection from z. Clearly $\pi(\tilde{G})$ is a planar embedding of G of the desired type.*

We denote the boundary of a face f, of a plane graph G by b(f). If G is connected, then b(f) can be regarded as a closed walk in which each cut edge of G in b(f) is traversed twice; when b(f) contains no cut edges, it is a cycle of G. For example, in the plane graph of figure 7,

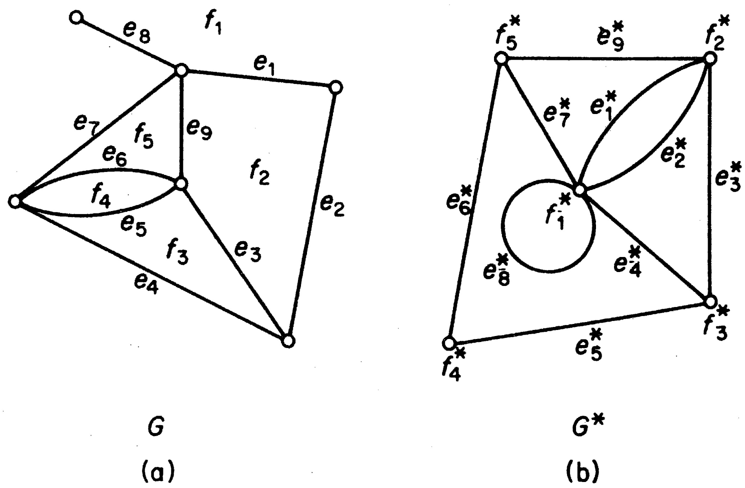
$$b(f_2) = v_l e_3 v_2 e_4 v_3 e_5 v_4 e_1 v_1$$

and

$$b(f_5) = v_7 e_{10} v_5 e_{11} v_8 e_{12} v_8 e_{11} v_5 e_8 v_6 e_9 v_7$$

A face f is said to be incident with th,e vertices and edges in its boundary. If e is a cut edge in a plane graph, just one face is incident with e; otherwise, there are two faces incident with e. We say that an edge separates the faces incident with it. The degree, d_G(f), of a face f is the number of edges with which it is incident (that is, the number of edges in b(f)), cut edges being counted twice. In figure 7, f_1 is incident with the vertices $v_t, v_3, v_4, v_5, v_6, v_7$ and the edges $e_1, e_2, e_5, e_6, e_7, e_9, e_{10}$; e_1 separates f_1 from f_2 and e_{11} separates f_5 from f_5; d(f_2)= 4 and d(f_5)=6.

Given a plane graph G, one can defile another graph G* as follows: corresponding to each face f of G. there is a vertex f* of G *, and corresponding to each edge e of G there is an edge e*of G*; two vertices f* and g* are joined by the edge e* in G* if and only if their corresponding faces f and g. are separated by the edge e in G. The graph G* is called the dual of G. A plane graph and its dual are shown in figures 8 (a) and 8 (b).



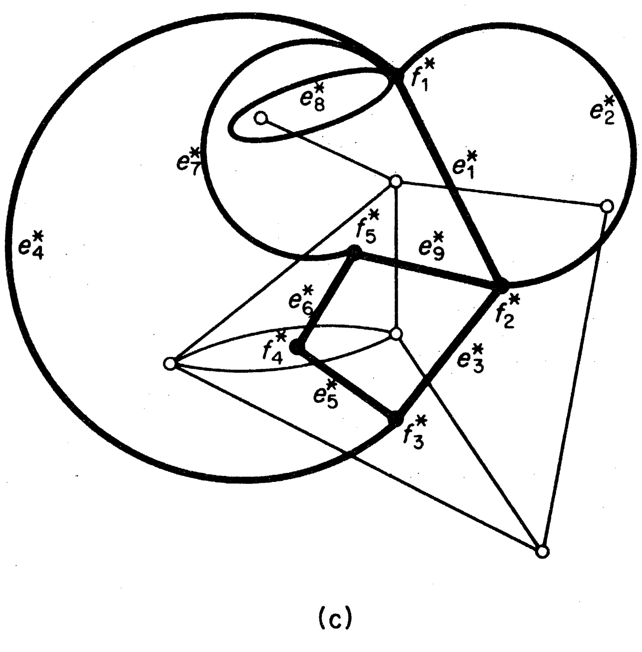


Figure 9.8: A plane graph and its dual

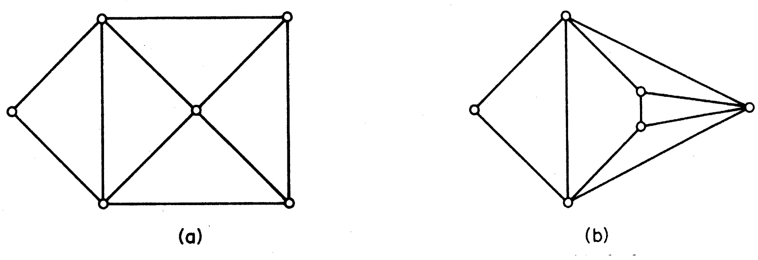


Figure 9.9: Isomorphic plane graphs with nonisomorp

It is easy to see that the dual G* of a plane graph G is planar; in fact, there is a natural way to embed G* in the plane. We place each vertex f* in the corresponding face f of G,and then draw each edge e* in such a way that it crosses the corresponding edge e of G exactly once (and crosses no other edge of G). This procedure is illustrated in figure 9.7C, where it is indicated by heavy points and lines. It is intuitively clear that we call always draw the dual as a plane graph in this way, but we shall float prove this fact. Note that if e is a loop of G, then e* is a cut edge of G*, and vice versa.

Although defined abstractly, it is sometimes convenient to regard the dual G* of a plane graph G as a plane graph (embedded as described above). One can then consider the dual G** of G*, and it is not difficult to prove that, when G is connected, G**≅ G; a glance at figure 8 (c) will indicate why this is so.

It should be noted that isomorphic plane graphs may have nonisomorphic duals. For example, the plane graphs in figure 9 are isomorphic, but their duals are not-the plane graph of figure 9 (a) has a face of degree five, whereas the plane graph of figure 9 (b) has no such face. Thus the notion of a dual is meaningful only for plane graphs, and cannot be extended to planar graphs in general.

The following relations are direct consequences of the definition of G*:

$$v(G^*) = \phi(G)$$

$$\varepsilon(G*) = \varepsilon(G)' \tag{9.1}$$

$$do.(f^*) = do(f)$$

for all f ∈ F(G).

Theorem 9.4 *Theorem 9.4 If G is a plane graph, then*

$$\sum_{f \in F} d(f) = 2\varepsilon$$

Proof 9.4 *Let G* be the dual of G. Then*

$$\begin{aligned}\sum_{f \in F(G)} d(f) &= \sum_{f^* \in V(G^*)} d(f^*)\\ &= 2\varepsilon(G^*)\\ &= 2\varepsilon(G)\end{aligned}$$

Exercises

1. (a) Show that a graph is planar if and only if each of its blocks is planar.
   (b) Deduce that a minimal nonplanar graph is a simple block.

2. A plane triangulation is a plane graph in which each face has degree three. Show that every simple plane graph is a spanning subgraph of some simple plane triangulation ($v \geq 3$).

3. Let G be a simple plane triangulation with $v \geq 4$. Show that G* is a simple 2 - edge - connected 3 - regular planar graph.

4. Show that any plane triangulation Gcontains a bipartite subgraph with 2e (G)/3 edges.
   (F. Harary, D. Matula)

9.3 EULER'S FORMULA

There is a simple formula relating the numbers of vertices, edges and faces in a connected plane graph. It is known as Euler's formula because Euler established it for those plane graphs defined by the vertices and edges of polyhedra.

Theorem 9.5 *IfG is a connected plane graph, then v-B+cP=2*

Proof 9.5 *By induction on ϕ, the number of faces of G. If ϕ = 1, then each edge of G is a cut edge and so G, being connected, is a tree. In this case ε = v -1, the* *theorem clearly holds. Suppose that it is true for all connected plane graphs with fewer than n faces, and let G be a connected plane graph with $n \geq 2$ faces. Choose an edge e of G that is not a cut edge. Then G - e is a connected plane graph and has n - 1 faces, since the two faces of G separated by e combine to form one face of G - e. By the induction hypothesis*

$$v(G-e) - \varepsilon(G-e) + \phi(G-e) = 2$$

and, using the relations

$$v(G-e) = v(G)$$

$$\varepsilon(G-e) = \varepsilon(G) - 1$$

$$\phi(G-e) = \phi(G) - 1$$

we obtain

$$v(G) - \varepsilon(G) + \phi(G) = 2 \tag{9.2}$$

The theorem follows by the principle of induction

Corollary 9.6 *All planar embeddings of a given connected planar graph have the same number of faces.*

Proof 9.6 *Let G and H be two planar embeddings of a given connected planar graph. Since $G \cong H$, $v(G) = v(H)$ and $\varepsilon(G) = \varepsilon(H)$. Applying theorem9.5, we have*

$$\phi(G) = \varepsilon(G) - v(G) + 2 = \varepsilon(H) - v(H) + 2 = \phi(H)$$

Corollary 9.7 *If G is a simple planar graph with $v \geq 3$, then $\varepsilon \leq 3v - 6$.*

Proof 9.7 *It clearly suffices to prove this for connected graphs. Let G be a simple connected graph with $v \geq 3$. Then $d(f) \geq 3$ for all $f \in F$, and*

$$\sum_{f \in F} d(f) \geq 3\phi$$

By theorem 9.4

$$2\varepsilon \geq 3\phi$$

Thus, from theorem 9.5

$$v - \varepsilon + 2\varepsilon/3 \geq 2$$

or

$$\varepsilon \leq 3v - 6$$

Corollary 9.8 *If G is a simple planar graph, then* $\delta \leq 5$.

Proof 9.8 *This is trivial for v = 1, 2. If v ≥ 3, then,*

$$\delta v \leq \sum_{v \in V} d(v) = 2\varepsilon \leq 6v - 12$$

It follows that 8≤5.

We have already seen that K_5 and $K_{3,3}$ are nooplanar. Here, we shall derive these two results as corollaries of theorem9.5.

Corollary 9.9 K_5 *is nonplanar.*

Proof 9.9

If K_5 were planar then" by corollary 9.7, we would have

$$10 = \varepsilon(K_5) < 3v(K_5) - 6 = 9$$

Thus Ks must be nonplanar

Corollary 9.10 $K_{3,3}$ *is nonplanar.*

Proof 9.10

Suppose that $K_{3,3}$ is planar and let G be a planar embedding of $K_{3,3}$. Since $K_{3,3}$ has no cycles of length less than four, every face of G must have degree at least four. Therefore, by theorem 9.4, we have

$$4\phi \leq \sum_{f \in F} d(f) = 2\varepsilon = 18$$

That is

$$\phi \leq 4$$

Theorem 9.5 now implies that

$$2 = v - \varepsilon + \phi \leq 6 - 9 + 4 = 1$$

which is absurd.

Exercises

1. Show that every planar graph is 6 - vertex - colourable.

Notes:

Chapter 10

BRIDGES

Unit - X

10.1 Bridges

In the study of planar graphs, certain subgraphs, called bridges, play an important role. We shall discuss properties of these subgraphs in this section.

Let H be a given subgraph of a graph G. We define a relation $\sim$ on E(G)\E(H) by the condition that $e_1 \sim e_2$ if there exists a walk W such that

1. the first and last edges of W are e_1 and e_2, respectively, and
2. W is internally-disjoint from H (that is, no internal vertex of W is a vertex of H).

It is easy to verify that $\sim$ is an equivalence relation on E(G)\E(H). A subgraph of G - E (H) induced by an equivalence class under the relation $\sim$ is called a bridge of H in G. It follows immediately from the definition that if B is a bridge of H, then B is a connected graph and, moreover, that any two vertices of B are connected by a path that is internally-disjoint from H. It is also easy to see that two bridges of H have no vertices in common except, possibly, for vertices of H. For a bridge B of H, we write V(B) $\cap$ V(H) = V(B, H), and call the vertices in this set the vertices of attachment of B to H. Figure 10.1 shows a variety of bridges of a cycle in a graph; edges of different bridges are represented by different kinds of lines.

In this section we are concerned with the study of bridges of a cycle C. Thus, to avoid

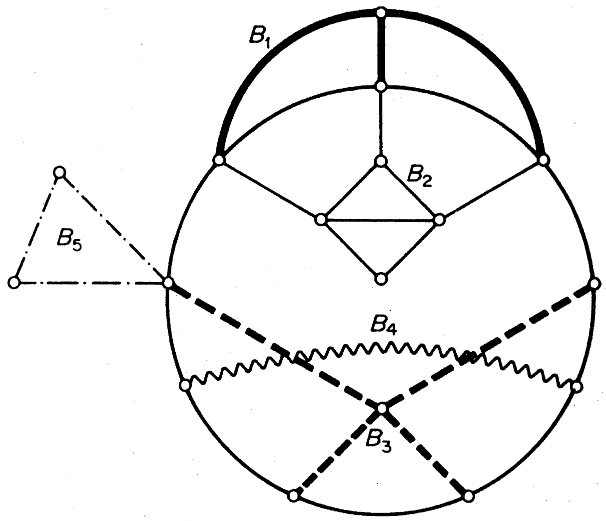


Figure 10.1: Bridges in a grap

repetition, we shall abbreviate 'bridge of C' to 'bridge' in the coming discussion; all bridges will be understood to be bridges of a given cycle C.

In a connected graph every bridge has at least one vertex of attachment, and in a block every bridge has at least two vertices of attachment. A bridge with k vertices of attachment is called a k-bridge. Two k-bridges with the same vertices of attachment are equivalent k-bridges; for example, in figure 10.1 B_1 and B_2 are equivalent 3-bridges.

The vertices of attachment of a k-bridge B with k $\geq$ 2 effect a partition of C into edge-disjoint paths, called the segments of B. Two bridge avoid one another if all the vertices of attachment of one bridge lie in a single segment of the other bridge; otherwise they overlap. In figure 10.1, B_2 and B_3 avoid one another, whereas B_1 and B_2 overlap. Two bridges B and B' are skew if there are four distinct vertices u, v, u' and v' of C such that u and v are vertices of attachment of B, u' and v' are vertices of attachment of B', and the four vertices appear in the cyclic order u, u', v, v' on C. In figure 10.1, B_3 and B_4 are skew, but B_1 and B_2 are not.

Theorem 10.1 *If two bridges overlap, then either they are skew or else they are equivalent 3 - bridges.*

Proof 10.1 *Suppose that the bridges B and B' overlap. Clearly, each must have at least two vertices of attachment. Now if either B or B' is a 2-bridge, it is easily* *verified that they must be skew. We may therefore assume that both B and B' have at least three vertices of attachment. There are two cases.*

Case 1 B and B' are not equivalent bridges. Then B' has a vertex of attachment u' between two consecutive vertices of attachment u and v of B. Since B and B' overlap, some vertex of attachment v' of B' does not lie in the segment of B connecting u and v. It now follows that B and B' are skew.

Case 2 B and B' are equivalent k-bridges, k $\geq$ 3. If k $\geq$ 4, then B and B' are clearly skew; if k = 3, they are equivalent 3 - bridges

Theorem 10.2 *If a bridge B has three vertices of attachment v_1, v_2 and v_3, then there exists a vertex v_0 in V(B)\V(C) and three paths P_1, P_2 and P_3 in B joining v_0 to v_1, v_2 and v_3, respectively, such that, for $i \neq j$, P_i and P_j have only the vertex v_0 in common (see figure 10.2).*

Proof 10.2 *Let P be a (v_1, v_2)-path in B, internally-disjoint from C. P must have an internal vertex v, since otherwise the bridge B would be just P, and would not contain a third vertex v_3. Let Q be a (v_3, v) - path in B; internally disjoint from C, and let v_0 be the first vertex of Q on P. Denote by P_1 the (v_0, v_1) - section of p^{-1}, by P_2 the (v_0, v_2) - section of P, and by P_3 the (v_0, v_3)-section of Q^{-1}. Clearly P_1, p_2 and P_3 satisfy the required conditions*

We shall now consider bridges in plane graphs. Suppose that G is a plane graph and that C is a cycle in G. Then C is a Jordan curve in the plane, and each edge of E(G)\E(C) is contained in one of the two regions Int C and Ext C. It follows that a bridge of C is contained entirely in Int C or Ext C. A bridge contained in Int C is called an inner bridge, and a bridge contained in Ext C, an outer bridge. In figure 9.11 B_1 and B_2 are inner bridges, and B_3 and B_4 are outer bridges.

10.2 KURATOWSKI'S THEOREM

Since planarity is such a fundamental property, it is clearly of importance to know which graphs are planar and which are not. We have already noted that, in particular, K_5 and $K_{3,3}$ are non-planar and that any proper subgraph of either of these graphs is planar. A remarkably simple characterization of planar graphs was given

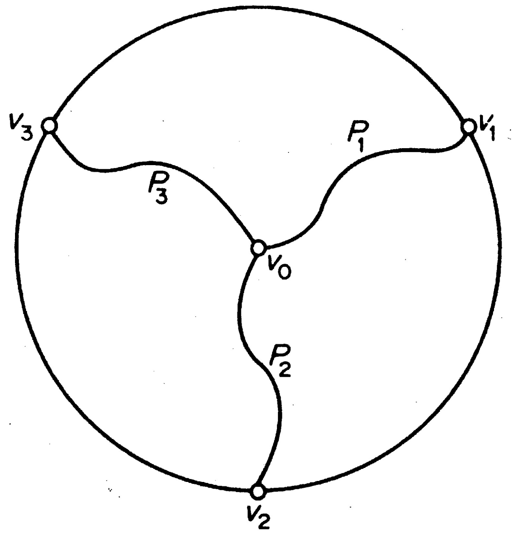


Figure 10.2:

by Kuratowski (1930). This section is devoted to a proof of Kuratowski's theorem. The following lemmas are simple observations, and we leave their proofs as an exercise (9.5.1).

Lemma 10.3 *If G is non-planar, then every subdivision of G is non-planar.*

Lemma 10.4 *If G is planar, then every subgraph of G is planar.*

Since K_5 and $K_{3,3}$ are non planar, we see from these two lemmas that if G is planar, then G cannot contain a subdivision of K_5 or of $K_{3,3}$. Kuratowski showed that this necessary condition is also sufficient.

Before proving Kuratowski's theorem, we need to establish two more simple lemmas.

Let G be a graph with a 2-vertex cut $\{u, v\}$. Then there exist edge-disjoint subgraphs G_1 and G_2 such that $V(G_1) \cap V(G_2) = u, v$ and $G_1 \cup G_2 = G$. Consider such a separation of G into subgraphs. In both G_1 and G_2 join u and v by a new edge e to obtain graphs H_1 and H_2. Clearly $G = (H_1 U H_2) - e$. It is also easily seen that $\varepsilon(H_i) < \varepsilon(G)$ for i = 1, 2.

Lemma 10.5 *If G is non planar, then at least one of H_1 and H_2 is also non planar.*

Proof 10.3 *By contradiction. Suppose that both H_1 and H_2 are planar. Let $\tilde{H}_1$ be a planar embedding of H_1, and let f be a face of $\tilde{H}_1$ incident with e. If $\tilde{H}_2$ is an* *embedding of H_2 in f such that $\tilde{H}_1$ and $\tilde{H}_2$ have only the vertices u and v and the edge e in common, then ($\tilde{H}_1 \cup \tilde{H}_2$) - e is a planar embedding of G. This contradicts the hypothesis that G is non planar.*

Lemma 10.6 *Let G be a non planar connected graph that contains no subdivision of K_5 or $K_{3,3}$ and has as few edges as possible. Then G is simple and 3 - connected.*

Proof 10.4 *By contradiction. Let G satisfy the hypotheses of the lemma. Then G is clearly a minimal non planar graph, and therefore must be a simple block. If G is not 3-connected, let $\{u,\ v\}$ be a 2 - vertex cut of G and let H_1 and H_2 be the graphs obtained from this ut as described above. By lemma 10.5, at least one of H_1 and H_2,say H_1, is non planar. Since $\varepsilon(H_1) < \varepsilon(G)$, H_1 must contain a subgraph K which is a subdivision of K_5 or $K_{3,3}$; moreover $K \nsubseteq G$, and so the edge e is in K. Let P be a (u, v)-path in H_2 - e. Then G contains the subgraph $(K \cup P) - e$, which is a subdivision of K and hence a subdivision of K_5 or $K_{3,3}$. This contradiction establishes the lemma*

We shall find it convenient to adopt the following notation in the proof of Kuratowski's theorem. Suppose that C isa cycle in a plane graph. Then we can regard the two possible orientations of C as 'clockwise' and 'anticlockwise'. For any two vertices, u and v of C, we shall denote by C[u, v] the (u, v)-path which follows the clockwise orientation of C; similarly we shall use the symbols C(u, v], C[u, v) and C(u, v)to denote the paths C[u, v] - u, C[u, v]-v and C[u, v]-{u, v}. We are now ready to prove Kuratowski's theorem. Our proof is based on that of Dirac and Schuster (1954).

Theorem 10.7 *A graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$.*

Proof 10.5 *We have already noted that the necessity follows from lemmas 9.10.1 and 9.10.2. We shall prove the sufficiency by contradiction.*
If possible, choose a nonplanar graph G that contains no subdivision of K_5 or $K_{3,3}$ and has as few edges as possible. From lemma 10.6 it follows that G is simple and 3-connected. Clearly G must also be a minimal nooplanar graph.

Let uv be an edge of G, and let H be a planar embedding of the planar graph G - uv. Since G is 3-connected, H is 2-connected and, by corollary 3.2.1, u and v are *contained together in a cycle of H. Choose a cycle C of H that contains u and v and i,s such that the number of edges in Int C is as large as possible.*

Since H is simple and 2-connected, each bridge of C in H must have at least two vertices of attachment. Now all outer bridges of C must be 2-bridges that overlap uv because, if some outer bridge were a k - bridge for $k \geq 3$ or a 2 - bridge that avoided uv, then there would be a cycle C' containing u and v with more edges in its interior than C, contradicting the choice of C.

In fact, all outer bridges of C in H must be single edges. For if a 2 - bridge with vertices of attachment x and y had a third vertex, the set x, y would be a 2 - vertex cut of G, contradicting the fact that G is 3 - connected.

No two inner bridges overlap. Therefore some inner bridge skew to uv must overlap some outer bridge. For otherwise, all such bridges could be transferred (one by one), and then the edge uv could be drawn in Int C to obtain a .planar embedding of G; since G is non planar, this is not possible. Therefore, there is an inner bridge B that is both skew to uv and skew to some outer bridge xy.

Two cases now arise, depending on whether B has a vertex of attachment different from u, v, x and y or not.

Case 1 *B has a vertex of attachment different from u, v, x and y. We can choose the notation so that B has a vertex of attachment v_1 in C(x, u). We consider two sub-cases, depending on whether B has a vertex of attachment in C(y, v) or not.*

Case 1a *B has a vertex of attachment v_2 in C(y, v). In this case there is a (v_1, v_2) - path P in B that is internally-disjoint from C. But then $(C \cup P) + \{uv, xy\}$ is a subdivision of $K_{3,3}$ in G, a contradiction.*

Case 1b *B has no vertex of attachment in C(y, v). Since B is skew to uv and to xy, B must have vertices of attachment v_2 in C(u, y] and v_3 in C[v, x). Thus B has three vertices of attachment v_1, v_2 and v_3. Then, there exists a vertex v_0 in $V(B)\setminus V(C)$ and three paths P_1, P_2 and P_3 in B joining v_0 to v_1, v_2 and v_3, respectively, such that, for $i \neq j$, P_i and P_j have only the vertex v_o in common. But now $(C \cup P_1 \cup P_2 \cup P_3) + \{uv, xy\}$ contains a subdivision of $K_{3,3}$, a contradiction. The subdivision of $K_{3,3}$ is indicated by, heavy lines.*

Case 2 *B has no vertex of attachment other than u, v, x and y. Since B is skew* *to both uv and xy, it follows that u, v, x and y must all be vertices of attachment of B. Therefore there exists a (u, v)-path P and an (x, y)-path Q in B such that (i) P and Q are internally-disjoint from C, and (ii)* $|V(P) \cap V(Q)| \geq 1$*. We consider two sub-cases, depending on whether P and Q have one or more vertices in common.*

Case 2a $|V(P) \cap V(Q)| \geq 1$*. In this case* $(C \cup P \cup Q) + \{uv, xy\}$ *is a sub, division of* K_5 *in G, again a contradiction.*

Case 2b $|V(P) \cap V(Q)| \geq 2$*. Let* u' *and* v' *be the first and last vertices of P on Q, and let* P_1 *and* P_2*. denote the* $(u, u')-$ *and* $(v', v)-$ *sections of P. Then* $(C \cup P_1 \cup P_2 \cup Q) + \{uv, xy\}$ *contains a subdivision of K3,3 in G, once more a contradiction.*

Thus all the possible cases lead to contradictions, and the proof is complete.

10.3 The Timetable Problem

Suppose in a school there are r teachers, T_1, T_2,..., T_r, and s classes, $C_1, C_2, .., C_s$. Each teacher T_i is expected to teach the class C_j for p_{ij} periods. It is clear that during any particular period, no more than one teacher can handle a particular class and no more than one class can be engaged by any teacher. Our aim is to draw up a timetable for the day that requires only the minimum number of periods. This problem is known as the "timetable problem".

To convert this problem into a graph - theoretic one, we form the bipartite graph $G = g(T, C)$ with bipartition (T, C), where T represents the set of teachers T_i and C represents the set of classes C_j. Further, T_i is made adjacent to C_j in G with p_{ij} edges iff teachers T_i is to handle class C_j for p_{ij} periods. Now, color the edges of G so that no two adjacent edges receive the same color. Then the edges in a particular color class, that is, the edges in that color form a matching in G and correspond to a schedule of work for a particular period. Hence, the minimum number of periods required is the minimum number of colors in an edge - coloring of G in which adjacent edges receive distinct colors; in other words, it is the edge - chromatic number of G. We now present these notions as formal definition.

Definition

An edge - coloring of a loopless graph G is a function $\pi : E(G) \to S$, where S is a set of distinct colors; it is proper if no two adjacent edges receive the same color. Thus a proper edge - coloring π of G is a function $\pi : E(G) \to S$ such that $\pi(e) \neq \pi(e')$

whenever edges e and e' are adjacent in G.

Definition

The minimum k for which a loopless graph G has a proper k - edge - coloring is called the edge chromatic number or chromatic index of G. It is denoted by $\chi'(G)$. G is k - edge - chromatic if $\chi'(G) = k$.

Theorem 10.8 *If G is a loopless bipartite graph, $\chi'(G) = \Delta(G)$.*

Proof 10.6 *The proof is by induction on the size (i.e., number of edges) m of G. The result is true for m = 1. Assume the result for bipartite graphs of size at most m - 1. Let G have m edges. Let $e = uv \in E(G)$. Then G-e has a proper Δ-edge-coloring, say c. Out of these Δ colors, suppose that, one particular color is not represented at both u and v. Then edge uv can be colored with this color and a proper Δ-edge-coloring of G is obtained.*

In the other case (i.e., in the case for which each of the Δ colors is represented either at u or at v) , since the degrees of u and v in G-e are at most Δ - 1, there exists a color out of the Δ colors that is not represented at u, and similarly there exists a color not represented at v. Thus, if color j is not represented at u in c, then j is represented at v in c, and if color i is not represented at v in c, then i is represented at u in c. Since G is bipartite and u and v are not in the same parts of the bipartition, there can exist no u-v path in G in which the colors alternate between i and j.

Notes:

Chapter 11

FIVE COLOR PROBLEM

Unit - XI

11.1 The Five Colour Theorem

Definition 11.1 *An assignment of colours to the vertices of a graph, there is no two adjacent vertices get the same colour is called* ***colouring*** *of a graph.*

Definition 11.2 *The vertices of a planar graph with atmost five colours is known as* ***five colour theorem****.*

Theorem 11.1 *Every planar graph is 5-colourable*

Proof 11.1 *We will prove the theorem by induction on the number of p points. For any planar graph having $p \leq 5$ points, the result is obvious since the graph is p-colourable.*

Now, let us assume that all planar graphs with p points is 5-colourable for some $p \geq 5$. Let G be a planar graph with $p + 1$ points. Then G has a vertex v of degree 5 or less. By induction hypothesis, the plane graph $G - v$ is 5-colourable. Consider a 5-colouring of a $G - v$ where $c_i, 1 \leq i \leq 5$, are the colours are used. If some colour, say c_j is not used in colouring vertices adjacent to v, then by assigning the colour c_j to v the 5 colouring of $G - v$ can be extended to a 5-colouring of G.

Hence, we have to consider only the case in which deg $v = 5$ and all the five colours are used for colouring the vertices of G adjacent to v.

Let v_1, v_2, v_3, v_4, v_5 be the vertices adjacent to v coloured c_1, c_2, c_3, c_4 and c_5 respectively and G_{13} denote the subgraph of $G - v$ induced by those vertices coloured c_1

or c_3. If v_1 and v_3 belong to different components of G_{13}, then 5-colouring of $G - r$ can be obtained by interchanging the colours of vertices in the component of G_{13} containing v_1. (Since no point of this component is adjacent to a point with colour c_1 or c_3 outside the component. this interchange of colours results in a coiluring of $G - v$). In this 5-colouring no vertex adjacent to v is coloured c_1, and hence by colouring v with c_1, a colouring of G is obtained.

If v_1 and v_3 are the same component of G_{13}, then in G there exists a $v_1 - v_3$ path all of whose points are colored c_1 or c_3. Hence there is no $v_2 - v_4$ path all whose points are colored c_2, c_4.

Hence, if G_{24} denotes the subgraph of $G - v$ induced by the points colored c_2 or c_4, then v_2 and v_4 belong to different components of G_{24}. Hence if we interchange the colors of the points in the component of G_{24} containing v_2, a new 5-coloring $G - v$ results and this, no point adjacent to v is colored c_2. Hence, by assigning colour c_2 to v, we can get a 5-coloring of G. This completes the induction and the proof.

11.2 Non-Hamiltonian Graph

Definition 11.3 *A spanning cycle in a graph is called a* ***Hamiltonian*** *cycle. A graph having a Hamiltonian cycle is called a* ***Hamiltonian graph***

Definition 11.4 *The* ***closure*** *of a graph G with p points is the graph obtained from G by repeatedly joining pairs of non adjacent vertices whose degree sum is at least p until no such pair remains. The closure of G is denoted by $c(G)$*

Theorem 11.2 *A graph is Hamiltonian iff its closure is hamiltonian*

Proof 11.2 *Let $x_1, x_2, \cdots, x_n$ be the sequence of edges added to G in obtaining $c(G)$. Let $G_1, G_2, \cdots, G_n = c(G)$ be the successive graphs obtained.*

G *is Hamiltonian* $\Leftrightarrow G_1$ *is Hamiltonian*
$\Leftrightarrow G_2$ *is Hamiltonian*
$\vdots$
$\Leftrightarrow G_n = c(G)$ *is Hamiltonian.*

Problem 11.2.1 *Show that the Petersen graph is Hamiltonian.*

Solution 11.2.1 *If the Petersen graph G has a Hamiltonian cycle C, then G-E(C) must be regular spanning subgraph of degree 1.*
Let us search for all 1-factors in G and show that none of them arise out a Hamiltonian cycle of G.
Case 1. *Consider the subset* $A = \{1a, 2b, 3c, 4d, 5e\}$ *of the edge set of G.*
Clearly A is a 1-factor of G, but G-A is the union of two disjoint cycles and hence is not a Hamiltonian cycle of G.
Case 2. *If the 1-factor contains 4 edges from A, then the only line passing through the remaining two points must also be included in the 1-factor, so that we again get A.*
Case 3. *If a 1-factor contains just 3 edges from A, then two such choices can be made.*
Sub-case 3A. *Let the one 1-factor contain 1a, 2b, and 3c. Now the subgraph induced by the remaining four points is a* P_4 *whose unique 1-factor is* $\{4d, 5e\}$. *Thus the 1-factor of G considered becomes A.*
Sub-case 3B*. Let the 1-factor contain 1a, 2b and 4d. Here again the remaining four points induce* P_4, *whose unique 1-factor is {3c, 5e }. Thus the 1-factor of G considered becomes A.*
Case 4. *If a 1-factor contains just 2 edges from A, then again two such choices are possible.*
Sub-case 4A. *Let the 1-factor contain 1a and 2b. In the subgraph induced by the remaining 6 points, point d has degree one and hence any 1-factor of that subgraph must contain edge 4d. Thus case 3 is repeated.*
Sub-case 4B. *Let the 1-factor contain 1a and 3b. In the subgraph induced by the remaining 6 points, point 2 has degree one and hence any 1-factor of that subgraph must contain edge 2b. Thus case 3 is repeated.*
Case 5. *Let a one factor contain just one edge of A, say 1a. If it a contains one more edge from A, then one of the earlier cases will be repeated. Hence we have choose the other four edges of this 1-factor from two paths, each of length 3. Hence the 1-factor is* $B = \{1a, ce, bd, 23, 45\}$. *Now G-B is again union of two disjoint cycles, and not a Hamiltonian cycle.*
Case 6. *Suppose there exists a 1-factor that does not contain any edge from A. It can contain at most two edges from the cycle* ***123451*** *and at most two edges from the cycle* ***acebda****. Hence it can contain at most four edges.*
Hence there does not exist such a 1-factor.
Since the above 6 cases cover all possible types of 1-factors, we see that G has no

1-factor arising out of a Hamiltonian cycle.
Hence, G has no hamiltonian cycle.
Thus, G is non-Hamiltonian.

Exercise 11.2.1 *Given an example of graph G such that $c(G)$ is not complete.*

Exercise 11.2.2 *Show that if G is a bipartite graph with an odd number of points, then G is non-Hamiltonian.*

Notes:

Chapter 12

DIRECTED GRAPHS & DIRECTED PATH

Unit - XII

12.1 Directed Graphs

Definition 12.1 *A* ***directed graph*** *D is a pair (V, A) where V is a finite nonempty set and A is a subset of* $V \times V - \{(x,x)/x \in V\}$. *The elements of V and A are respectively are called vertices and arcs. If* $(u,v) \in A$ *then the arc (u, v) is said to have u as its initial vertex and v as its terminal vertex. Also the arc (u,v) is said to join u to v.*

Theorem 12.1 *In a graph D, sum of the in-degrees of all the vertices is equal to the sum of their out degrees, each sum being equal to the number of arcs in D.*

Proof 12.1 *Let q denote the number of arcs in* $D = (V, A)$.
Let $B = \sum_{v \in V} d^+(v)$ *and* $C = \sum_{v \in V} d^-(v)$.
An arc (u,w) contributes one to the out-degree of u and one to the in-degree of w.
Hence each arc contributes 1 to the sum B and 1 to the sum C.
Hence, $B = C = q$.

Definition 12.2 *A* ***walk*** *in a digraph is a finite alternating sequence* $W = v_0x_1v_1, \cdots, x_nv_n$ *of vertices and arcs in which* $x_i = (v_{i-1}, v_i)$ *for every arc* x_i. *W is called a walk from* v_0 *to* v_n *or a* $v_0 - v_n$ *walk. The vertices* v_0 *and* v_n *are called the* ***origin*** *and* ***terminus*** *of W respectively and* $v_1, v_2, \cdots, v_{n-1}$ *are called its* ***internal vertices***.

*The **length** of a walk is the number of occurrence of arcs in it. A walk in which the origin and terminus coincide is called a **closed walk**.*

12.2 Directed Paths and Cycles

Definition 12.3 *A **path** is a walk in which all the vertices are distinct. A **cycle** is a nontrivial closed walk whose origin and internal vertices are distinct.*
*If there is a path from u to v is said to be **reachable** from u. A digraph is called **strongly connected** or **connected** or **strong** if every pair of points are mutually reachable. A digraph is called **unilaterally connected** or **unilateral** if for every pair of points, at least one is reachable from the other. A digraph is called **weakly connected** or **weak** if the underlying graph is connected. A digraph is called **disconnected** if the underlying graph is disconnected.*

Theorem 12.2 *The edges of a connected graph $G = (V, E)$ can be oriented so that the resulting digraph is strongly connected iff every edge of G is contained in at least one cycle.*

Proof 12.2 *Suppose the edges of G can be oriented so that the resulting digraph becomes strongly connected.*
If possible, let $e = vw$ be an edge of G not lying on any cycle. Now, as soon as e is oriented, one of the vertices u and w becomes non-reachable from the other. Hence, an orientation of the required type is not possible, giving contradiction. Hence every edge of G lies on a cycle.
Conversely, let every edge of G lie on a cycle.
Let $S = v_1, v_2, \cdots, v_n, v_1$ be a cycle in G. Orient the edges of S so that S becomes a directed cycle and hence becomes a strongly connected sub-digraph. If $V = \{v_1, \cdots, v_n\}$ then we are through. Otherwise, let w be a vertex of G not in S such that w is adjacent to a vertex v_i of S. Let $e = v_i w$. By hypothesis e lies on some cycle C. We choose a direction of C and give the orientation determined by this direction to the edges of C which are not already oriented. The resulting enlarged oriented graph is also a strongly connected as it can be got from S by a sequence of additions of simple directed paths. (For example, if $v \in S$ and u is a point on a simple directed $v_i - v_j$ path P added to S then in the enlarged oriented graph the $u - v_j$ sub-path of P followed by the $v_j - v$ sub-path of S give a directed $u - v$ path. Also the $v - v_j$ sub-path of S followed by the $v_i - u$ sub-path of P give a directed $v - u$ path. This

type of argument can be repeated for each addition og simple, directed paths.)
This process can be repeated till we get a strongly connected oriented spanning subgraph of G. The remaining edges can now be oriented in any way. The resulting oriented graph is strongly connected. This completes the proof.

Exercise 12.2.1 *Show that every Eulerian graph is strongly connected and prove its converse not true.*

Notes:

Chapter 13

NETWORKS

Unit - XIII

13.1 Flows

A network N is a digraph D (the underlying digraph of N) with two distinguished subsets of vertices, X and Y and a non-negative integer valued function c defined on its arc set A; the sets X and Y are assumed to be disjoint and nonempty.
We represent a network by drawing its underlying digraph and labeling each arc with its capacity. Then the below digram (13.1) shows that the network with two sources x_1 and x_2, three sinks y_1, y_2andy_3 and four intermediate vertices's v_1, v_2, v_3 and v_4.
If $S \subseteq V$, we denote $V \backslash S$ by $\overline{S}$. If f is a real-valued function defined on the arc set of A of N, and if $K \subseteq A$, we denote $\sum\limits_{a \in K} f(a)$ by $f(K)$. Furthermore, if K is a set of arcs of the form $(S, \overline{S})$, we shall write $f^+(S)$ for $f(S, \overline{S})$ and $f^-(S)$ for $f(S, \overline{S})$.
A flow in a network N is an integer-valued f defined on A such that

$$0 \leq f(a) \leq c(a) \quad \text{for all} \quad a \in A \tag{13.1}$$

and

$$f^-(v) = f^+(v) \quad \text{for all} \quad v \in I \tag{13.2}$$

The value of $f(a)$ of f on an arc a can be likened to the rate at which material is transported along a under the flow f. The upper bound in condition (13.1) is called the *capacity constraint*; it imposes the natural restriction that the rate of flow along an arc cannot exceed the capacity of the arc. Condition (13.2) , is called *conservation condition*, requires that, for any intermediate vertex v, the rate at which material

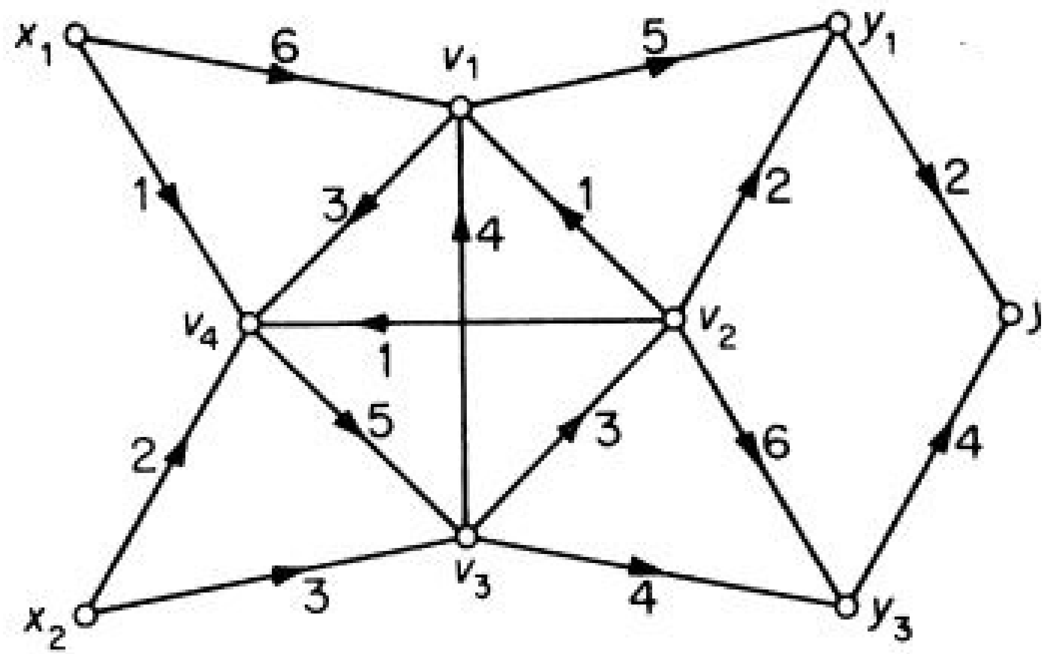


Figure 13.1: A Network

is transported into v is equal to the rate at which it is transported out of v. Note that every network has at least one flow, since the function f defined by $f(a) = 0$, for all $a \in A$, clearly satisfies both (13.1) and (13.2) and; it is called *zero flow*. A less trivial example of a flow is given in figure (13.2). The flow along each arc is indicated in bold type.

If S is a subset of vertices's in a network N and f is a flow in N, then $f^+(S) - f^(S)$ is called the *resultant flow* of out of S, and $f^-(S) - f^+(S)$ the *resultant flow* into S, relative to f. Since the conservation condition requires that the resultant flow out of X is equal to the resultant flow into Y. This common quantity is called the *value* of f, and is denoted by $valf$; thus

$$valf = f^+(X) - f^-(X)$$

The value of the flow indicated in figure(13.2) is 6. A flow f in N is a *maximum flow* if there is no flow f' in N such that val $f' > valf$. Such flows are of obvious importance in the context of transportation networks. The problem of determining a maximum flow in an arbitrary network can be reduced to the case of networks that have just one source and one sink by means of a simple device. Given a network N, construct a new network N' as follows:

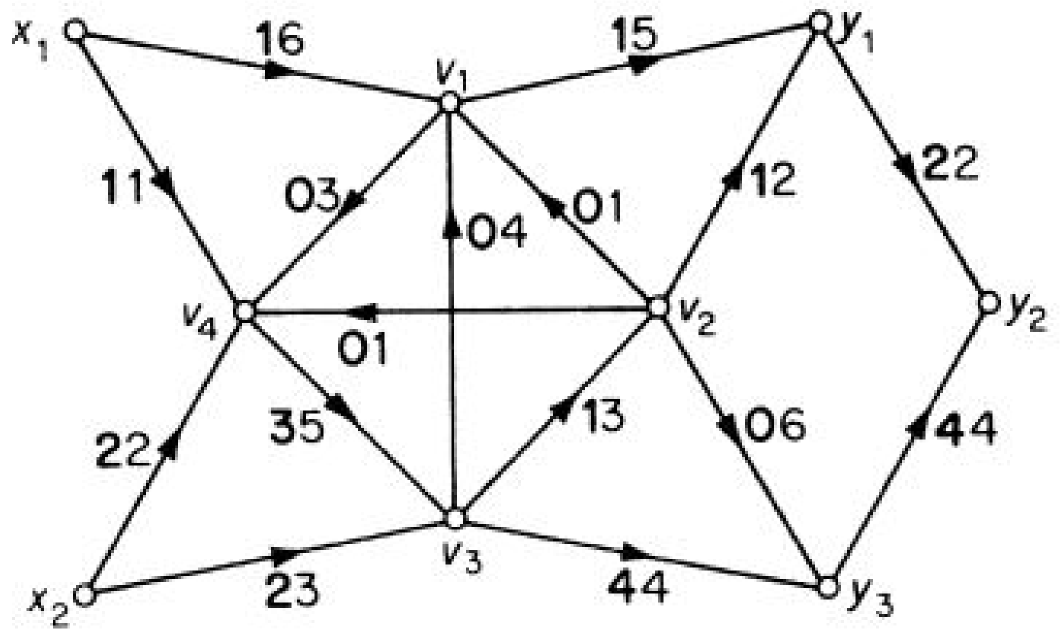


Figure 13.2: A flow in network

(i) adjoin two new vertices's x and y to N;

(ii) join x to each vertex in X by an arc of capacity ∞;

(iii) join each vertex in Y to y by an arc of capacity ∞;

(iv) designate x as the source and y as the sink of N'

Figure (13.3) illustrates this procedure as applied to the network N of figure (13.1). Flows in N' and N correspond to one another in a simple way. If f is a flow in N such that the resultant flow out of each source and into each sink is non-negative (it suffices to restrict our attention to such flows) then the function f' defined by

$$f'(a) = \begin{cases} f(a), & if \quad \text{a is an arc of } N \\ f^{+}(v) - f^{-}(v) & if \quad a = (x, v) \\ f^{-}(v) - f^{+}(v) & if \quad a = (v, y) \end{cases} \tag{13.3}$$

is a flow N' such that $valf' = valf$. Conversely, the restriction to the arc set of N of a flow in N' is a flow in N having the same value. Therefore, throughout the next three sections, we shall confirm our attention to networks that have a single source x and a single sink y.

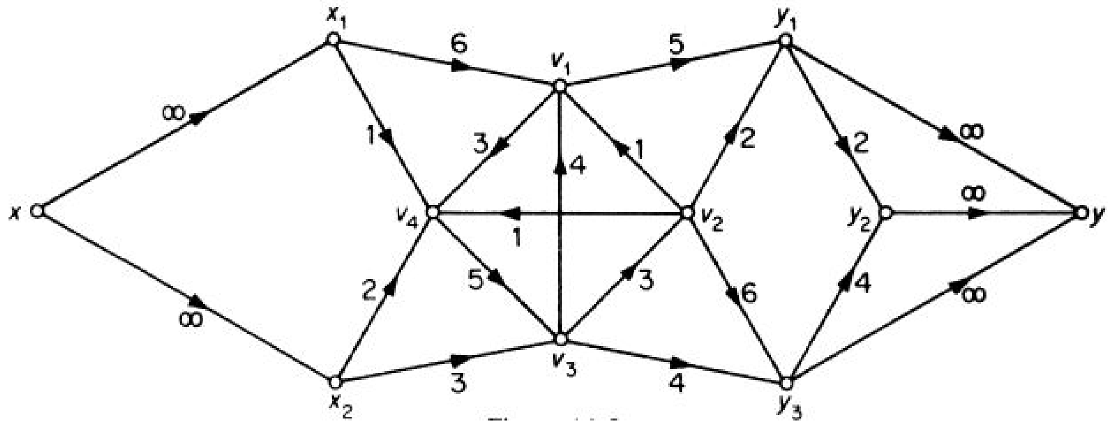


Figure 13.3:

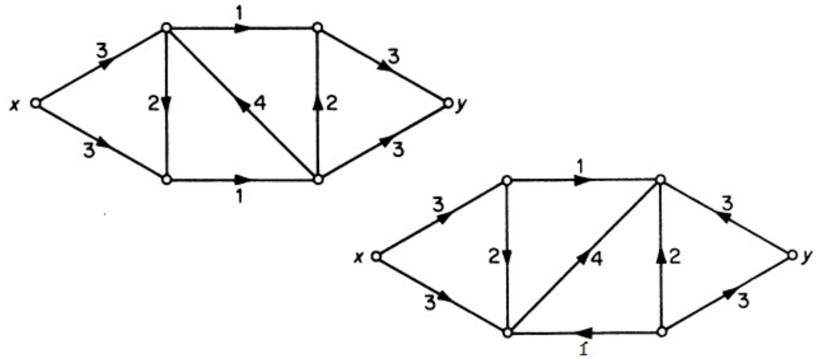


Figure 13.4: Exercise: 1

Exercise

(1) For each of the following networks , determine all possible flows and the value of a maximum flow.

(2) Show that, for any flow f in N and any $S \subseteq V$,

$$\sum (f^+(v) - f^-(v)) = f^+(S) - f^-(S)$$

(Note that, in general, $\sum f^+(v) \neq f^+(S)$and $\sum f^-(v) \neq f^-(S)$)

(3) Show that, relative to any flow f in N, the resultant flow out of X is equal to the resultant flow into Y.

(4) Show that

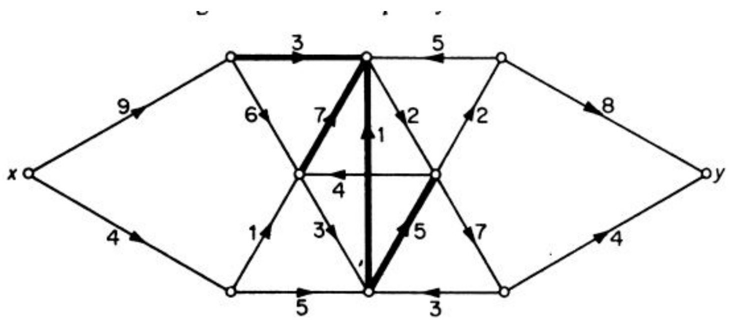


Figure 13.5: A cut in a network

(a) the function f' given by Equ.(13.3)is a flow in N' and that $valf' = valf$;

(b) the restriction to the arc set of N of a flow in N having the same value.

13.2 Cuts

Let N be a network with a single source x and a single sink y. A cut in N is a set of arcs of the form $(S, \overline{S})$, where $x \in S$ and $y \in \overline{S}$. In the network of figure(13.5), a cut is indicated by heavy lines.

The capacity of a cut K is the sum of the capacities of its arcs. We denote the capacity of K by cap K; thus $capK = \sum_{a \in K} c(a)$ The cut indicated in figure(13.5) has capacity 16.

Lemma: For any flow f and any cut $(S, \overline{s})$ in N

$$valf = f^{+}(S) - f^{-}(S)$$

Proof Let f be a flow and $(S, \overline{S})$ a cut in N. From the definitions of flow and value of a flow, we have

$$f^{+}(v) - f^{-}(v) = \begin{cases} valf & if \quad v = x \\ 0 & if \quad v \in S \backslash \{x\} \end{cases}$$

Summing these equations over S and simplifying, we obtain

$$valf = \sum (f^{+}(v) - f^{-}(v)) = f^{+}(S) - f^{-}(S)$$

It is convenient to call an arc a *f-zero* if $f(a) = 0$, f-positive if $f(a) > 0$, *f-unsaturated* if $f(a) < c(a)$ and f-saturated if $f(a) = c(a)$.

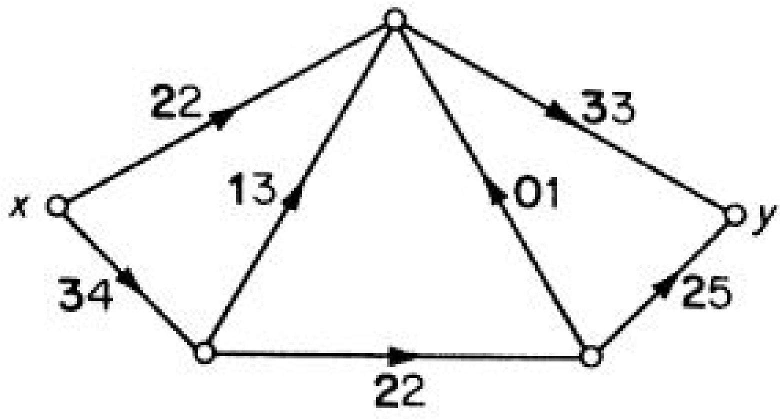


Figure 13.6: Exercise: 1

Theorem: For any flow f and any cut $K = (S, \overline{S})$ in$N$

$$valf \leq capK$$

Furthermore, the above equality holds if each arc in $(S, \overline{S})$ is f-saturated and each arc in $(S, \overline{S})$ is f-zero.

Note: A cut K in N is a maximum cut if there is no cut K' in N such that $capK' < capK$.. If f^* is a maximum flow and $\tilde{K}$ is a minimum cut, we have, as a special case of theorem, that

$$valf^* \leq cap\tilde{K}$$

Corollary: Let f be a flow and K be a cut such that $valf = capK$. Then f is a maximum flow and K is a minimum cut.

Exercise

(1) In the above network,

(a) determine all cuts;

(b) find the capacity of a minimum cut;

(c) show that the flow indicated is a maximum flow.

(2) Show that, if there exists no directed $(x, y)-$ path in N, then the value of a maximum flow and the capacity of a minimum cut are both zero.

Notes:

Chapter 14

MAX-FLOW MIN-CUT THEOREM

Unit - XIV

14.1 The Max-Flow Min-Cut theorem

Let f be a flow in a network N. With each path P in N we associate a non-negative integer $\upsilon(p) = min\upsilon(a)$
where

$$\upsilon(a) = \begin{cases} c(a) - f(a) & \text{if } a \text{ is a forward arc of } P \\ f(a) & \text{if } a \text{ is a reverse arc of } P \end{cases}$$

As may be easily be seen, $\upsilon(P)$ is the largest amount by which the flow along P can be increased (relative to f) without violating condition Equ(13.1). The path P is said to be f-saturated if $\upsilon(P) > 0$(or, equivalently, if each forward arc of P is f-unsaturated and each reverse arc of P is f-positive). Put simply, an f-unsaturated path is one that is not being used to its full capacity. An f-incrementing path is an f-unsaturated path from the source x to the sink y.
The existence of an f-incrementing path P in a network is significant since it implies that f is not a maximum flow; in fact; by sending an additional flow of $\upsilon(P)$ along P, one obtains a new flow $\hat{f}$ is defined by

$$\hat{f}(a) = \begin{cases} f(a) + \upsilon(P) & \text{if a is a forward arc of P} \\ f(a) - \upsilon(P) & \text{if a is a reverse arc of P} \\ f(a) & \text{otherwise.} \end{cases}$$

for which $val\hat{f} = valf + \upsilon(P)$. We shall refer to $\hat{f}$ as the revised flow based on P.

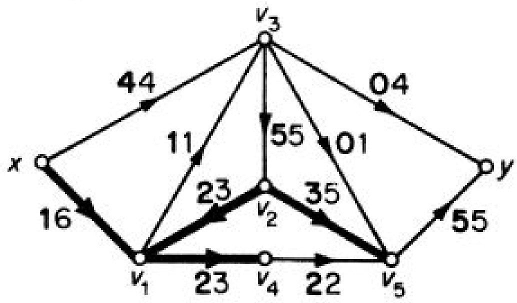


Figure 14.1: An f-unsaturated tree

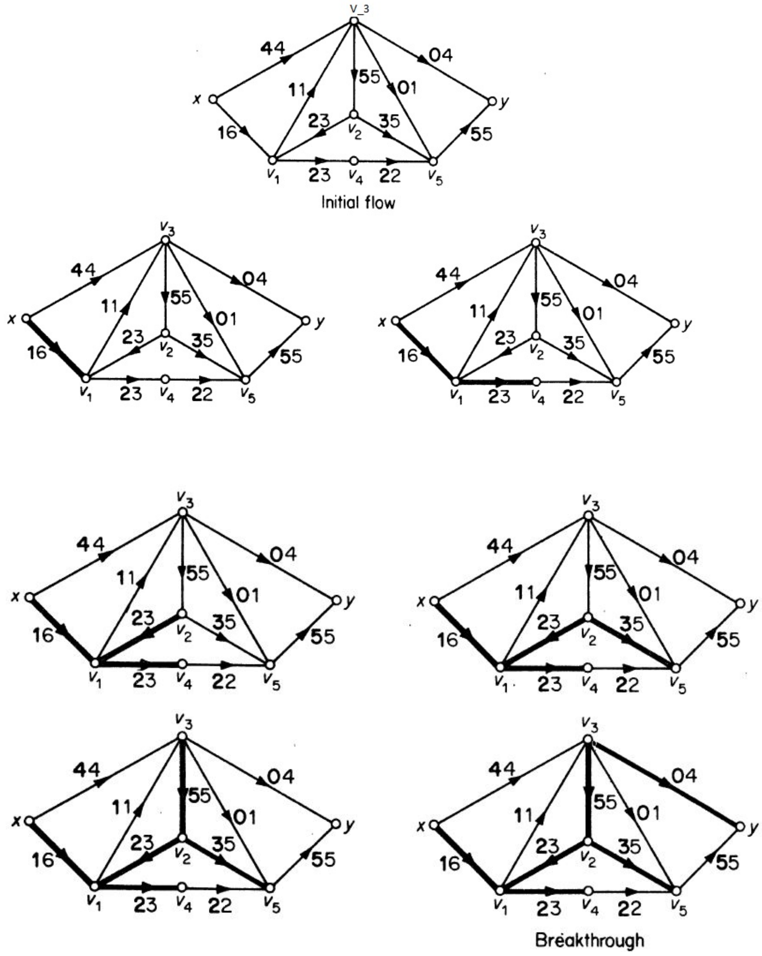
Theorem: A flow f in N is a maximum flow if and only if N contains no f-incrementing path.

Theorem: (Max-flow min-cut theorem) In any network, the value of a maximum flow is equal to the capacity of minimum cut.

Proof: It is of central importance in graph theory. Many results on graphs turn out to be easy consequences of this theorem as applied to suitably chosen networks. We prove this theorem by finding algorithm for a maximum flow in a network. It is also known as *labeling method*. Starting with a known flow, for instance the zero flow, it recursively constructs a sequence of flows of increasing value, and terminates with a maximum flow. After the construction of each new flow f, a subroutine called the *labeling procedure* is used to find an f-incrementing path, if one exists. If such a path P is found, then , the revised flow based on P, is constructed and taken as the next flow in the sequence. If there is no such path, the algorithm terminates; then by the above theorem f is a maximum flow.

To describe the labeling procedure we need the following definition. A tree T in N is an f-unsaturated tree if (i) $x \in V(T)$, and (ii) for every vertex v of T, the unique (x, v) path in T is an f-unsaturated path. Such a tree is shown in figure.
The search of an f-incrementing path involves growing an f-unsaturated tree T in N. Initially, T consists of just the source x. At any stage, there are two ways in which the tree may grow:

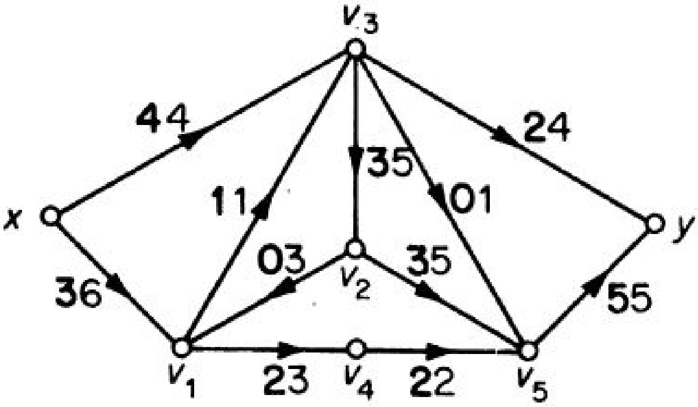
- If there exists an f-unsaturated arc a in $(S, \overline{S})$, where $S = V(T)$, then both a and its head are adjoined to T.



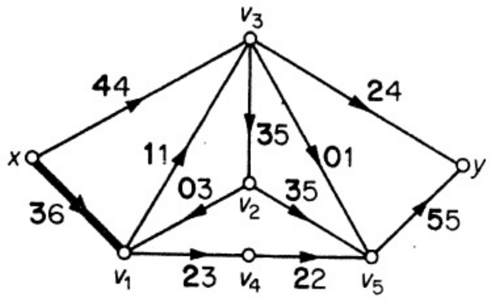
- If there exists an f-positive arc a in $(S, \overline{S})$, then both a and its tail are adjoined to T.

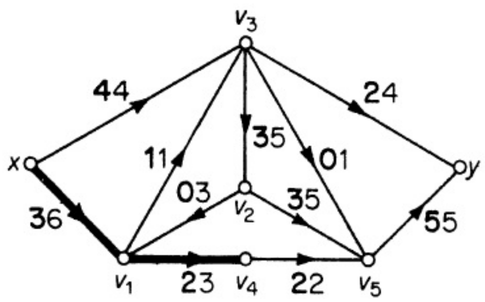
Clearly, each of the above procedures results in an enlarged f-unsaturated tree. Now either T eventually reaches the sink y or it stops growing before reaching y. The former case is referred to as *breakthrough*; in the event of breakthrough, the (x, y)- path in T is our desired f-incrementing path. If, however, T stops growing before reaching y, we deduce from theorem and corollary that f is a maximum flow. In figure two iterations of this tree-growing procedure are illustrated. The first leads to breakthrough;the second shows that the resulting revised flow is a maximum flow.

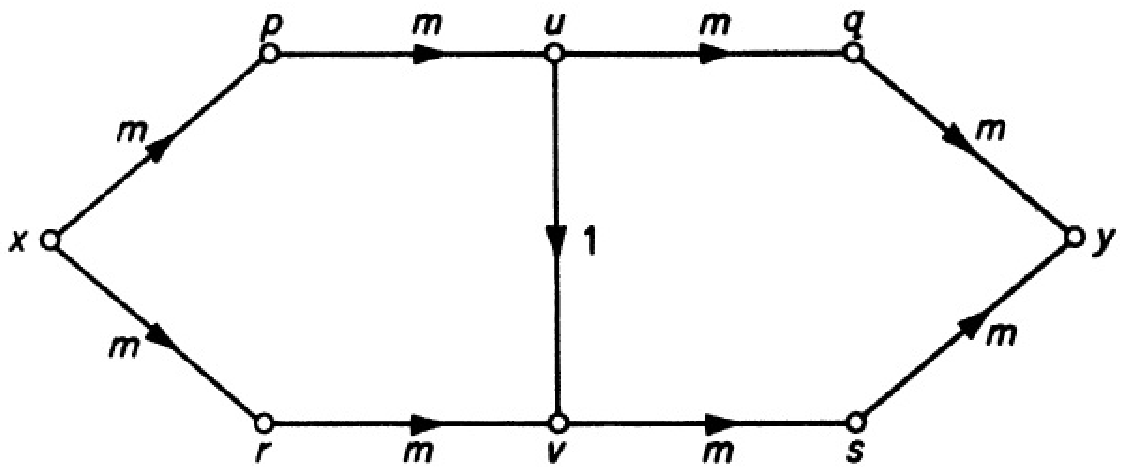
The labeling procedure is a systematic way of growing an f-unsaturated tree T. In the process of growing T, it assigns to each vertex v of T the label $l(v) = \upsilon(P_v)$, where P_v is unique (x, v)-path in T. The advantage of this labeling is that, in the event of breakthrough, we not only have the f-incrementing path P_y, but also the



Revised flow







quantity $v(P_y)$ with which to calculate the revised flow based on P_y.The labeling procedure begins by assigning to the source x the label $l(x) = \infty$. It continues according to the following rules:

- If a ia an f-unsaturated arc whose tail u is already labeled but whose head v is not, then v is labeled $l(v) = min\{l(u), c(a) - f(a)\}$
- If a is an ff-positive arc whose head u is already labeled but whose tail v is not, then v is labeled $l(v) = min\{l(u), f(a)\}$.

In each of the above case, v is said to be labeled based on u. To scan a labeled vertex u is to label all unlabeled vertices that can be labeled based on u. The labeling procedure is continued until either the sink y is labeled(breakthrough) or all labeled vertices have been scanned and no mire vertices can be labeled (implying that f is a maximum flow).

Consider, for example, the network N in figure(14.1), Clearly, the value of a maximum flow is N is $2m$. The labeling method will use the labeling procedure $2m + 1$ times if it starts with the zero flow and alternate between selecting $xpuvsy$ and $xrvuqy$ as an incrementing path; for, in each case, the flow value increases by exactly one. Since m is arbitrary, the number of computational steps required to implement the labeling method in this instance can be bounded by no function of v ans ε. In other words, it is not good algorithm. The refinement suggested as follows: in the labeling procedure, scan on a 'first-labeled first-scanned' basis; that is, before scanning a labeled vertex u, scan the vertices that were labeled before u. It can be seen that this amounts to selecting a shortest incrementing path. With this refinement clearly, the maximum flow in the network of figure(14.1) would be found in just two iterations of the labeling procedure.

Exercise

1. Show that, in any network N (with integer capacities), there is a maximum flow f such that $f(a)$ is an integer for all $a \in A$.

2. Consider a network N such that with each arc a is associated an integer $b(a) \leq c(a)$. Modify the labeling method to find a maximum flow f in N subject to the constraint $f(a) \geq b(a)$ for all $a \in A$ (assuming that there is an initial flow satisfying this condition).

14.2 Applications

14.2.1 MENGER'S THEOREMS

Lemma: Let N be a network with the source x and sink y in which each arc has unit capacity. Then

(a) the value of a maximum flow in N is equal to the maximum number m of arc-disjoint directed (x, y)-paths in N; and

(b) the capacity of a minimum cut in N is equal to the minimum number n of arcs whose deletion destroys all directed (x, y)-paths in N.

Theorem: Let $x$and y be two vertices of a digraph D. Then the maximum number of arc-disjoint directed (x, y)-paths in D is equal to the minimum number of arcs whose deletion destroys all directed (x, y)-paths in D.

Theorem: Let x and y be two vertices of a graph G. Then the maximum number of edge-disjoint (x, y)- paths in G is equal to the minimum number of edges whose deletion destroys all (x, y)-paths in G.

Corollary: A graph$G$ is $k-$ edge connected if and only if any two distinct vertices of G are connected by at least k edges-disjoint paths.

Theorem: Let x and y be two vertices of a digraph D, such that x is not joined to y. Then the maximum number of internally-disjoint directed (x, y) -paths in D is equal to the minimum number of vertices whose deletion destroys all directed (x, y)-paths in D.

Proof: Construct a new digraph $D^{'}$ from D as follows:

- split each vertex $v \in V\backslash\{x, y\}$ into two new vertices $v^{'}$ and $v^{"}$, and join them by an arc $(v, v^{'})$;
- replace each arc of D with head $v \in V\backslash\{x, y\}$ by new arc with head $v^{'}$, and each arc of D with tail $v \in V\backslash\{x, y\}$ by a new arc with tail $v^{"}$. This construction is illustrated in figure.

Now to each directed (x, y) path in $D^{'}$ there corresponds a directed (x, y)-paths in D obtained by contracting all arcs of type $(c^{'}, v^{"})$; and, conversely to each directed (x, y)-path in D, there corresponds a directed (x, y)-path in $D^{'}$obtained by splitting each internal vertex of the path.Furthermore, two directed (x, y)-paths in $D^{'}$ are arc-disjoint if and only if the corresponding paths in D are internally-disjoint. It follows that the maximum number of arc-disjoint directed(x, y)-paths in $D^{'}$ is equal to the maximum number of internally -disjoint directed(x, y)-paths in D. Similarly, the minimum number of arcs in $D^{'}$ whose deletion destroys all directed (x, y)-paths is equal to the minimum number of vertices's in D whose deletion destroys all directed (x, y)-paths.

Theorem: Let$x$ and y be two non adjacent vertices's of a graph G. Then the maximum number of internally-disjoint (x, y)-paths in G is equal to the minimum number of vertices's whose deletion destroys all (x, y)-paths.

Corollary: A graph G with $v \geq k + 1$ is k-connected if and only if any two distinct vertices's of G are connected by at least k internally disjoint paths.

Exercise

1. Let G be a graph and let S and T be two disjoint subsets of V, Show that the maximum number of vertex-disjoint paths with one end in S and one end in T is equal to the minimum number of vertices's whose deletion separated S from T (that is, after deletion no component contains a vertex of S and a vertex of T)

2. Show that if G is k-connected with $k \geq 2$, then any k vertices's of G are contained together in some cycle.

Notes:

DISTANCE EDUCATION - CBCS

MODEL QUESTION PAPER

M.Sc., DEGREE EXAMINATION, NOVEMBER 2019

Mathematics

GRAPH THEORY

(2018-2019 onwards)

Time: 3 hours **Maximum: 75 Marks**

PART A **(10×2=20)**

Answer **all** questions.

1. Define simple graph.
2. Define tree.
3. What is block?
4. Explain shortly in Ramsay's numbers.
5. If G is bipartite, then $\chi' = \Delta + 1$.
6. Prove that every critical graph is a block.
7. If G is planar, prove that every subgraph of G is planar.
8. Define coloring.
9. What is network?
10. Define cuts.

PART B **(5×5=25)**

Answer **all** questions choosing either (a) or (b).

11. (a) Prove that in any group of n persons($n \geq 2$), there are at least two with the same number of friends.

(Or)

(b) If $\delta \geq 2$, then show that G contains a cycle.

12. (a) Show that in a tree, any path of maximum length contains the center of the tree.

(Or)

(b) Prove that a set S is an independent set of G if and only if $V \setminus S$ is a covering of G.

13. (a) Let G be a connected graph that is not an odd cycle. Then G has a 2-edge colouring in which both colours are represented at each vertex of degree at least two.

(Or)

(b) Prove that a graph G is embeddable in the plane if and only if it is embeddable on the sphere.

14. (a) If G is a loopless bipartite graph, prove that $\chi'(G) = \Delta(G)$.

(Or)

(b) Prove that every planar graph is 5-colourable.

15. (a) For any flow f and any cut $(S, \overline{s})$ in N, prove that

$$val f = f^{+}(S) - f^{-}(S)$$

(Or)

(b) Write the applications of Max-flow and min-cut theorem.

PART C **(3×10=30)**

Answer any **three** questions.

16. i) Show that in a graph, the number of edges common to a cycle and an edge cut is even.
ii) Give an example of a graph with n vertices and $n-1$ edges that is not a tree.

17. Prove that a matching M in G is a maximum matching if and only if G contains no $M-$augmenting path.

18. State and Prove Brook's theorem.

19. Prove that the edges of a connected graph $G = (V, E)$ can be oriented so that the resulting digraph is strongly connected iff every edge of G is contained in at least one cycle.

20. In any network, prove that the value of a maximum flow is equal to the capacity of minimum cut.

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